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Distributional representations and dominance of a Lévy process over its maximal jump processes

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Distributional identities for a Lévy process X_t , its quadratic variation process V_t and its maximal jump processes, are derived, and used to make “small time” (as $t \downarrow 0$) asymptotic comparisons between them. The representations are constructed using properties of the underlying Poisson point process of the jumps of X . Apart from providing insight into the connections between X , V , and their maximal jump processes, they enable investigation of a great variety of limiting behaviours. As an application, we study “self-normalised” versions of X_t , that is, X_t after division by $\sup_{0 < s \leq t} \Delta X_s$, or by $\sup_{0 < s \leq t} |\Delta X_s|$. Thus, we obtain necessary and sufficient conditions for $X_t / \sup_{0 < s \leq t} \Delta X_s$ and $X_t / \sup_{0 < s \leq t} |\Delta X_s|$ to converge in probability to 1, or to ∞ , as $t \downarrow 0$, so that X is either comparable to, or dominates, its largest jump. The former situation tends to occur when the singularity at 0 of the Lévy measure of X is fairly mild (its tail is slowly varying at 0), while the latter situation is related to the relative stability or attraction to normality of X at 0 (a steeper singularity at 0). An important component in the analyses is the way the largest positive and negative jumps interact with each other. Analogous “large time” (as $t \rightarrow \infty$) versions of the results can also be obtained.

Keywords: Distributional representation; domain of attraction to normality; dominance; Lévy process; maximal jump process; relative stability

1. Introduction

We study relations between a Lévy process $X = (X_t)_{t \geq 0}$, its quadratic variation process $V = (V_t)_{t \geq 0}$ and its maximal jump processes, with particular interest in how these processes, and how positive and negative parts of the X process, interact. Representations of distributions related to these processes are calculated and used as a basis for making asymptotic (small time) comparisons in their behaviours.

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A convenient way of proceeding is to derive identities for the distributions of X_t modified by subtracting a number of its largest jumps, or its jumps of largest modulus, up until time t , joint with V_t , modified similarly. These identities are obtained by considering the Poisson point process of jumps of X , allowing for possible ties in the order statistics of the jumps.

The distributions thus obtained enable the study of a wide variety of small or large time kinds of behaviour of X . As an application, we investigate “self-normalised” versions of X_t , giving a comprehensive analysis of the behaviour of $X_t/\sup_{0 < s \leq t} \Delta X_s$ and $X_t/\sup_{0 < s \leq t} |\Delta X_s|$ as $t \downarrow 0$, and similarly with X_t replaced by $|X_t|$. Two extreme situations are considered; first, when X is of comparable size to a maximal jump process, for example, $X_t/\sup_{0 < s \leq t} |\Delta X_s| \xrightarrow{P} 1$ as $t \downarrow 0$; or, alternatively, when X dominates a maximal jump process, in the sense that $X_t/\sup_{0 < s \leq t} |\Delta X_s| \xrightarrow{P} \infty$ as $t \downarrow 0$; and similarly with X_t replaced by $|X_t|$, and/or $|\Delta X_s|$ replaced by ΔX_s . Complementary to these is the way the largest positive and negative jumps interact with each other.

Such results can be seen as continuations in one way or another of a growing literature in this area which has some classical antecedents. The original developments occurred in the context of random walks, where the concept of “trimming” by removing extremes from a sample sum has been studied extensively in the past. Our particular emphasis on the ratio of the process to its extremes goes back in the random walk situation to results of Darling [10] and Arov and Bobrov [2]. Later, Maller and Resnick [41] gave conditions for a random walk to be comparable in magnitude to its large values (a heavy-tailed situation), while Kesten and Maller [24, 25] studied the other end of the spectrum, when the sum dominates its large values (see Table 1 of [25] for a convenient summary).

Subsequent to these papers there was much development in the general area of trimmed sums, especially concerning heavy tailed distributions; see, for example, Csörgő, Haeusler and Mason [8], Berkes and Horváth [3], Berkes, Horváth and Schauer [4], and Griffin and Pruitt [21]. We mention in this context also results of Silvestrov and Teugels [48] concerning sums and maxima of random walks and triangular arrays, and Ladoucette and Teugels [31] for an insurance application. There are also recent results about the St. Petersburg game; Gut and Martin-Löf [22] give a “maxtrimmed” version of the game, while Fukker, Györfi and Kevei [18] determine the limit distribution of the St. Petersburg sum conditioned on its maximum. Csörgő and Simons [9] give a review of the later St. Petersburg literature.

For almost sure versions of particular kinds of sum/max relationships, see Feller [16], Kesten and Maller [26] and Pruitt [43].

Studies of small time or local behaviour of Lévy processes go back to the work of Lévy and Khintchine [28, 29], in the 1930s. More recent work, relevant to our topic, includes that of Doney [11], who gives conditions for a Lévy process X to remain positive near 0 with probability approaching 1, and Andrew [1], who similarly analyses the behaviours of the positive and negative jump processes near 0. There is a connection also with results of Bertoin [6], who in studying regularity of a Lévy process X at 0 was concerned with the dominance of the positive part of X over its negative part, when X is of bounded variation. For further background along these lines, we refer to Doney [12].

Despite all this activity, there seems to have been little done so far by way of relating the Lévy process directly to its large jumps, as we do herein. Of course, our methods rely substantially on previously developed foundational work. Our representations of the trimmed Lévy process, for example, are inspired by those of LePage [32, 33], LePage, Woodroffe and Zinn [34] and Mori [42] for trimmed sums via order statistics, and Khintchine's [29] inverse Lévy measure method. (The corresponding representations are incorporated in our Lemma 1.) In another direction, Rosiński [46] collects a number of alternative series representations for Lévy processes, especially with a view to simulation of the process.

Our paper is organised as follows. The dominance results are in Sections 3 and 5. Section 4 compares the positive and negative jump processes. Before this, in Section 2, we set up notation and, in Theorem 2.1, derive the distribution identities using the Poisson point process structure of the jumps. Section 2 also recalls some basic facts concerning Poisson point processes and constructs the distribution of the relevant Poisson random measure from the jumps of X . Particular attention is paid to the possibility of tied jumps, related to atoms in the canonical measure of X . We make brief mention of some other possible applications of the methodology in the final discussion Section 6.

2. Distributional representations

Our object of study will be a real-valued Lévy process $X = (X_t)_{t \geq 0}$ with canonical triplet (γ, σ^2, Π) , thus having characteristic function $Ee^{i\theta X_t} = e^{t\Psi(\theta)}$, $t \geq 0$, $\theta \in \mathbb{R}$, with characteristic exponent

$$\Psi(\theta) := i\theta\gamma - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}_*} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}}) \Pi(dx). \quad (2.1)$$

Here, $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a Lévy measure on \mathbb{R} , that is, a Borel measure on $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}_*} (x^2 \wedge 1) \Pi(dx) < \infty$. Define measures $\Pi^{(+)}$, $\Pi^{(-)}$, and $\Pi^{|\cdot|}$ on $(0, \infty)$ such that $\Pi^{(+)}$ is Π restricted to $(0, \infty)$, $\Pi^{(-)}$ is $\Pi(-\cdot)$ restricted to $(0, \infty)$, and $\Pi^{|\cdot|} := \Pi^{(+)} + \Pi^{(-)}$. The positive, negative and two-sided tails of Π are

$$\begin{aligned} \bar{\Pi}^+(x) &:= \Pi\{(x, \infty)\}, & \bar{\Pi}^-(x) &:= \Pi\{(-\infty, -x)\} \quad \text{and} \\ \bar{\Pi}(x) &:= \bar{\Pi}^+(x) + \bar{\Pi}^-(x), & x &> 0. \end{aligned} \quad (2.2)$$

We are only interested in small time behaviour of X_t , so we eliminate trivial cases by assuming $\bar{\Pi}(0+) = \infty$ or $\bar{\Pi}^+(0+) = \infty$, as appropriate. Let $\Delta\Pi(y) := \Pi(\{y\})$, $y \in \mathbb{R}_*$, and $\Delta\bar{\Pi}(y) := \bar{\Pi}(y-) - \bar{\Pi}(y)$, $y > 0$. Denote the jump process of X by $(\Delta X_t)_{t \geq 0}$, where $\Delta X_t = X_t - X_{t-}$, $t > 0$, with $\Delta X_0 \equiv 0$. The quadratic variation process associated with X is

$$V_t := \sigma^2 t + \sum_{0 < s \leq t} (\Delta X_s)^2, \quad t > 0,$$

with $V_0 \equiv 0$. Recall that X is of bounded variation if $\sum_{0 < s \leq t} |\Delta X_s| < \infty$ a.s. for all $t > 0$, equivalently, if $\sigma^2 = 0$ and $\int_{|x| \leq 1} |x| \Pi(dx) < \infty$. If this is the case, (2.1) takes the form

$$i\theta d_X + \int_{\mathbb{R}} (e^{i\theta x} - 1) \Pi(dx),$$

where d_X is the drift of X .

In deriving representations for the joint distributions of X_t , V_t and the r th maximal jump processes, it is convenient to work with the processes having the r largest jumps, or the r jumps largest in modulus, subtracted. These “trimmed” processes are no longer Lévy processes, but we can give useful representations for their marginal distributions. The expressions are in terms of a truncated Lévy process, together with one or two Poisson processes, and a Gamma random variable, all processes and random variables independent of one another.

For any integer $r = 1, 2, \dots$, let $\Delta X_t^{(r)}$ and $\widetilde{\Delta X}_t^{(r)}$ be the r th largest positive jump and the r th largest jump in modulus up to time t , respectively. Formal definitions of these, allowing for the possibility of tied values (we choose the order uniformly among the ties), are given in Section 2.1 below. “One-sided” and “modulus” trimmed versions of X are then defined as

$${}^{(r)}X_t := X_t - \sum_{i=1}^r \Delta X_t^{(i)} \quad \text{and} \quad {}^{(r)}\widetilde{X}_t := X_t - \sum_{i=1}^r \widetilde{\Delta X}_t^{(i)}, \quad (2.3)$$

with corresponding trimmed quadratic variation processes

$${}^{(r)}V_t := V_t - \sum_{i=1}^r (\Delta X_t^{(i)})^2 \quad \text{and} \quad {}^{(r)}\widetilde{V}_t := V_t - \sum_{i=1}^r (\widetilde{\Delta X}_t^{(i)})^2, \quad t > 0.$$

Recall the definitions of the tails of Π in (2.2). Let

$$\overline{\Pi}^{\leftarrow}(x) = \inf\{y > 0 : \overline{\Pi}(y) \leq x\}, \quad x > 0,$$

be the right-continuous inverse of the nonincreasing function $\overline{\Pi}$, and similarly for $\overline{\Pi}^{+, \leftarrow}$ and $\overline{\Pi}^{-, \leftarrow}$. By convention, the inf of the empty set is taken as ∞ . The following properties of the inverse function will be used frequently (see Resnick [45], Section 0.2). For each $x, y > 0$, $\overline{\Pi}^{\leftarrow}(x) \leq y$ if and only if $\overline{\Pi}(y) \leq x$; $\overline{\Pi}(\overline{\Pi}^{\leftarrow}(x)) \leq x \leq \overline{\Pi}(\overline{\Pi}^{\leftarrow}(x)-)$; and $\overline{\Pi}^{\leftarrow}(\overline{\Pi}(x)) \leq x$; similarly, for $\overline{\Pi}^{\pm}$. We refer to Appendix A in Fan [15] for more details.

We introduce four families of processes, indexed by $v > 0$, truncating jumps from sample paths of X_t and V_t , respectively. Let $v, t > 0$. When $\overline{\Pi}(0+) = \infty$, we set

$$\begin{aligned} \widetilde{X}_t^v &:= X_t - \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| \geq \overline{\Pi}^{\leftarrow}(v)\}} \quad \text{and} \\ \widetilde{V}_t^v &:= V_t - \sum_{0 < s \leq t} (\Delta X_s)^2 \mathbf{1}_{\{|\Delta X_s| \geq \overline{\Pi}^{\leftarrow}(v)\}}. \end{aligned} \quad (2.4)$$

When $\bar{\Pi}^+(0+) = \infty$, we set

$$X_t^v := X_t - \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s \geq \bar{\Pi}^{+, \leftarrow}(v)\}} \quad \text{and} \quad V_t^v := V_t - \sum_{0 < s \leq t} (\Delta X_s)^2 \mathbf{1}_{\{\Delta X_s \geq \bar{\Pi}^{+, \leftarrow}(v)\}}.$$

Under the assumptions $\bar{\Pi}(0+) = \infty$ and $\bar{\Pi}^+(0+) = \infty$, $(\tilde{X}_t^v)_{t \geq 0}$ and $(X_t^v)_{t \geq 0}$ are well-defined Lévy processes with canonical triplets, respectively,

$$\left(\gamma - \mathbf{1}_{\{\bar{\Pi}^{\leftarrow}(v) \leq 1\}} \int_{\bar{\Pi}^{\leftarrow}(v) \leq |x| \leq 1} x \Pi(dx), \sigma^2, \Pi(dx) \mathbf{1}_{\{|x| < \bar{\Pi}^{\leftarrow}(v)\}} \right) \quad (2.5)$$

and

$$\left(\gamma - \mathbf{1}_{\{\bar{\Pi}^{+, \leftarrow}(v) \leq 1\}} \int_{\bar{\Pi}^{+, \leftarrow}(v) \leq x \leq 1} x \Pi(dx), \sigma^2, \Pi(dx) \mathbf{1}_{\{x < \bar{\Pi}^{+, \leftarrow}(v)\}} \right). \quad (2.6)$$

Our main result in this section gives very general representations for the joint distributions of $((^{(r)}\tilde{X}_t, (^{(r)}\tilde{V}_t, |\widetilde{\Delta X}_t^{(r)}|))$ and of $((^{(r)}X_t, (^{(r)}V_t, \Delta X_t^{(r)}))$, allowing for possible tied values in the large jumps. We make the convention throughout that a Poisson random variable with parameter 0 is 0. Note that then the expressions in (2.7), (2.8) and (2.10) below are zero when Π has no atoms. But we do not assume this.

Theorem 2.1. *Let $r \in \mathbb{N} = \{1, 2, 3, \dots\}$ and \mathfrak{S}_r be a $\text{Gamma}(r, 1)$ random variable. Suppose $Y^\pm = (Y_t^\pm)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ are independent Poisson processes with $EY_1^\pm = EY_1 = 1$. Assume that X , \mathfrak{S}_r , Y^+ , Y^- , and Y are independent as random elements.*

(i) Assume $\bar{\Pi}(0+) = \infty$. For each $v > 0$, let

$$\kappa^\pm(v) := (\bar{\Pi}(\bar{\Pi}^{\leftarrow}(v)-) - v) \frac{\Delta \Pi(\pm \bar{\Pi}^{\leftarrow}(v))}{\Delta \bar{\Pi}(\bar{\Pi}^{\leftarrow}(v))} \mathbf{1}_{\{\Delta \bar{\Pi}(\bar{\Pi}^{\leftarrow}(v)) \neq 0\}} \quad (2.7)$$

and for $v > 0$, $t > 0$, set

$$\tilde{G}_t^v := \bar{\Pi}^{\leftarrow}(v)(Y_{t\kappa^+(v)}^+ - Y_{t\kappa^-(v)}^-) \quad \text{and} \quad \tilde{H}_t^v := (\bar{\Pi}^{\leftarrow}(v))^2(Y_{t\kappa^+(v)}^+ + Y_{t\kappa^-(v)}^-). \quad (2.8)$$

Then, for each $t > 0$, we have

$$(^{(r)}\tilde{X}_t, (^{(r)}\tilde{V}_t, |\widetilde{\Delta X}_t^{(r)}|) \stackrel{\mathbb{D}}{=} (\tilde{X}_t^v + \tilde{G}_t^v, \tilde{V}_t^v + \tilde{H}_t^v, \bar{\Pi}^{\leftarrow}(v))|_{v=\mathfrak{S}_r/t}. \quad (2.9)$$

(ii) Assume $\bar{\Pi}^+(0+) = \infty$. For each $v > 0$, let $\kappa(v) := \bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v)-) - v$, and for $v > 0$, $t > 0$, set

$$G_t^v := \bar{\Pi}^{+, \leftarrow}(v)Y_{t\kappa(v)} \quad \text{and} \quad H_t^v := (\bar{\Pi}^{+, \leftarrow}(v))^2Y_{t\kappa(v)}. \quad (2.10)$$

Then, for each $t > 0$, we have

$$(^{(r)}X_t, (^{(r)}V_t, \Delta X_t^{(r)}) \stackrel{\mathbb{D}}{=} (X_t^v + G_t^v, V_t^v + H_t^v, \bar{\Pi}^{+, \leftarrow}(v))|_{v=\mathfrak{S}_r/t}. \quad (2.11)$$

Remark 2.1. Processes $^{(r)}\tilde{X}_t$ and $^{(r)}X_t$ are not Lévy processes; their increments are not independent, or homogeneous in distribution. But the identities (2.9) and (2.11) express their marginal distributions in terms of distributions of Lévy processes, mixed in a sense according to their r th largest jumps, with allowance made for ties. This opens the possibility for results obtained from analyses of the underlying Lévy processes to be transferred to the trimmed processes. We exemplify this procedure in a variety of ways in Sections 3 and 5.

As an immediate corollary of Theorem 2.1, the following identities will be useful.

Corollary 1. *Using the notation in Theorem 2.1, we have, for $x \in \mathbb{R}$, $y \geq 0$, $t > 0$, $r = 1, 2, \dots$:*

(i) *when $\bar{\Pi}(0+) = \infty$,*

$$\begin{aligned} P(^{(r)}\tilde{X}_t \leq x | \Delta \tilde{X}_t^{(r)}, ^{(r)}\tilde{V}_t \leq y | \Delta \tilde{X}_t^{(r)})^2 \\ = \int_0^\infty P(\tilde{X}_t^v + \tilde{G}_t^v \leq x \bar{\Pi}^\leftarrow(v), \tilde{V}_t^v + \tilde{H}_t^v \leq y (\bar{\Pi}^\leftarrow(v))^2) P(\mathfrak{S}_r \in t dv); \end{aligned} \quad (2.12)$$

(ii) *when $\bar{\Pi}^+(0+) = \infty$,*

$$\begin{aligned} P(^{(r)}X_t \leq x \Delta X_t^{(r)}, ^{(r)}V_t \leq y (\Delta X_t^{(r)})^2) \\ = \int_0^\infty P(X_t^v + G_t^v \leq x \bar{\Pi}^{+, \leftarrow}(v), V_t^v + H_t^v \leq y (\bar{\Pi}^{+, \leftarrow}(v))^2) P(\mathfrak{S}_r \in t dv). \end{aligned} \quad (2.13)$$

In proving Theorem 2.1, we make use of the underlying Poisson point process (PPP) structure of the jumps of a Lévy process. We begin in Section 2.1 with a precise definition of the order statistics of a PPP when tied values may be present. In Section 2.2, we review basic properties of standard PPPs and in Section 2.3 construct the distribution of a Poisson random measure (PRM) from the jumps of a Lévy process through a series of marking and deterministic transformations. Also, in Section 2.3, we derive the joint distribution of the trimmed point process using the point process order statistics. This machinery allows us to complete the proof of Theorem 2.1 in Section 2.4.

2.1. Order statistics with ties

Introduce \mathbb{X} as the point measure associated with the jumps of X :

$$\mathbb{X} = \sum_s \delta_{(s, \Delta X_s)}.$$

\mathbb{X} is a Poisson point process¹ (PPP) on $[0, \infty) \times \mathbb{R}_*$ with intensity measure $ds \otimes \Pi(dx)$. Analogously, the PPPs of positive and negative jumps and jumps in modulus associated with \mathbb{X} are

$$\begin{aligned}\mathbb{X}^+ &= \sum_s \mathbf{1}_{(0, \infty)}(\Delta X_s) \delta_{(s, \Delta X_s)}, & \mathbb{X}^- &= \sum_s \mathbf{1}_{(0, \infty)}(-\Delta X_s) \delta_{(s, -\Delta X_s)}, \\ \mathbb{X}^{|\cdot|} &= \mathbb{X}^+ + \mathbb{X}^- = \sum_s \delta_{(s, |\Delta X_s|)},\end{aligned}$$

having intensity measures $ds \otimes \Pi^{\pm, |\cdot|}(dx)$, respectively. For $t > 0$, we consider restrictions of these processes to the time interval $[0, t]$ by introducing

$$\mathbb{X}_t(\cdot) := \mathbb{X}([0, t] \times \mathbb{R}_* \cap \cdot) \quad \text{and} \quad \mathbb{X}_t^{\pm, |\cdot|}(\cdot) = \mathbb{X}^{\pm, |\cdot|}([0, t] \times (0, \infty) \cap \cdot).$$

Assume $\bar{\Pi}(0+) = \infty$ and $t > 0$. Our first task is to specify the points with maximum modulus in \mathbb{X}_t .

Let $\tilde{T}^{(1)}(\mathbb{X}_t)$ be randomly chosen, independently of $(X_t)_{t \geq 0}$, according to the discrete uniform distribution in the set $\{0 \leq s \leq t : |\Delta X_s| = \sup_{0 \leq u \leq t} |\Delta X_u|\}$, which is almost surely finite. Then define $\widetilde{\Delta X}_t^{(1)} = \widetilde{\Delta X}^{(1)}(\mathbb{X}_t) := \Delta X_{\tilde{T}^{(1)}(\mathbb{X}_t)}$. Define the maximum modulus trimmed point process on $[0, t] \times \mathbb{R}_*$ by

$$^{(1)}\widetilde{\mathbb{X}}_t := \mathbb{X}_t - \delta_{(\tilde{T}^{(1)}(\mathbb{X}_t), \widetilde{\Delta X}_t^{(1)})}.$$

Let $r = 2, 3, \dots$. Iteratively, we define $\tilde{T}^{(r)}(\mathbb{X}_t) := \tilde{T}^{(1)}(^{(r-1)}\widetilde{\mathbb{X}}_t)$ and $\widetilde{\Delta X}_t^{(r)} := \Delta X_{\tilde{T}^{(r)}(\mathbb{X}_t)}$. The r -fold modulus trimmed point process of modulus jumps is then defined by

$$^{(r)}\widetilde{\mathbb{X}}_t := \mathbb{X}_t - \sum_{i=1}^r \delta_{(\tilde{T}^{(i)}(\mathbb{X}_t), \widetilde{\Delta X}_t^{(i)})}.$$

In a similar way, under the assumption $\bar{\Pi}^+(0+) = \infty$, we can define the ordered pairs

$$(T^{(1)}(\mathbb{X}_t^+), \Delta X_t^{(1)}), (T^{(2)}(\mathbb{X}_t^+), \Delta X_t^{(2)}), (T^{(3)}(\mathbb{X}_t^+), \Delta X_t^{(3)}), \dots \in [0, t] \times (0, \infty),$$

such that $\Delta X_t^{(1)} \geq \dots \geq \Delta X_t^{(r)}$ are the r th largest order statistics of positive jumps of X sampled on time interval $[0, t]$. By subtracting the points corresponding to large jumps, analogously as we did for $^{(r)}\widetilde{\mathbb{X}}_t$, we then define the r -positive trimmed point process of positive jumps by

$$^{(r)}\mathbb{X}_t^+ := \mathbb{X}_t^+ - \sum_{1 \leq i \leq r} \delta_{(T^{(i)}(\mathbb{X}_t^+), \Delta X_t^{(i)})}.$$

¹For necessary material on point processes, we refer to Chapter 12 in Kallenberg [23] or Chapter 5 in Resnick [44].

2.2. Standard Poisson point process

In this section, we provide alternative constructions of $\mathbb{X}_t, {}^{(r)}\widetilde{\mathbb{X}}_t, \mathbb{X}_t^+, {}^{(r)}\mathbb{X}_t^+$, this time starting from homogeneous processes.

Let (\mathfrak{U}_i) , (\mathfrak{U}'_i) and (\mathfrak{E}_i) be independent, where (\mathfrak{U}_i) and (\mathfrak{U}'_i) are i.i.d. sequences of uniformly distributed random variables in $(0, 1)$, and (\mathfrak{E}_i) is an i.i.d. sequence of exponentially distributed random variables with common parameter $E\mathfrak{E}_i = 1$. Then $\mathfrak{S}_r = \sum_{i=1}^r \mathfrak{E}_i$ is a $\text{Gamma}(r, 1)$ random variable, $r \in \mathbb{N}$.

For $t > 0$, we introduce

$$\mathbb{V}_t := \sum_{i \geq 1} \delta_{(t\mathfrak{U}_i, \mathfrak{S}_i/t)} \quad \text{and} \quad \mathbb{V}'_t := \sum_{i \geq 1} \delta_{(t\mathfrak{U}_i, \mathfrak{U}'_i, \mathfrak{S}_i/t)}.$$

Then \mathbb{V}_t and \mathbb{V}'_t are homogeneous PPPs on $[0, t] \times (0, \infty)$ and $[0, t] \times (0, 1) \times (0, \infty)$ with intensity measures $ds \otimes dv$ and $ds \otimes du' \otimes dv$, respectively. For $r \in \mathbb{N} := \{0, 1, 2, \dots\}$, we define their r -fold trimmed counterparts by

$${}^{(r)}\mathbb{V}_t := \sum_{i > r} \delta_{(t\mathfrak{U}_i, \mathfrak{S}_i/t)} \quad \text{and} \quad {}^{(r)}\mathbb{V}'_t := \sum_{i > r} \delta_{(t\mathfrak{U}_i, \mathfrak{U}'_i, \mathfrak{S}_i/t)}.$$

When $\overline{\Pi}(0+) = \infty$, we consider the transformation

$$(I, I, \overline{\Pi}^{\leftarrow}) : [0, t] \times (0, 1) \times (0, \infty) \rightarrow [0, t] \times (0, 1) \times (0, \infty), \quad (s, u', v) \mapsto (s, u', \overline{\Pi}^{\leftarrow}(v)).$$

Still assuming $\overline{\Pi}(0+) = \infty$, by the Radon–Nikodym theorem, there exist Borelian functions $g^\pm : (0, \infty) \rightarrow (0, \infty)$ with $g^+ + g^- \equiv 1$ such that $d\Pi^\pm = g^\pm d\Pi^{|\cdot|}$ and, in particular,

$$\overline{\Pi}^\pm(x) = \int_{(x, \infty)} g^\pm(y) \Pi^{|\cdot|}(dy), \quad x > 0. \quad (2.14)$$

We use g^+ to return the sign to the process by a second transformation $m : [0, t] \times (0, 1) \times (0, \infty) \rightarrow [0, t] \times \mathbb{R}_*$, defined by

$$m(s, u', x) := \begin{cases} (s, x), & \text{if } u' < g^+(x), \\ (s, -x), & \text{if } u' \geq g^+(x). \end{cases} \quad (2.15)$$

In summary, let $\mathbb{V}_t^{m \circ (I, I, \overline{\Pi}^{\leftarrow})}$ be the point process on $[0, t] \times \mathbb{R}^*$, being the image of the composition of the above transformations applied to \mathbb{V}'_t :

$$\begin{aligned} \mathbb{V}_t &\xrightarrow{(I, I, \overline{\Pi}^{\leftarrow})} \mathbb{V}'_t^{(I, I, \overline{\Pi}^{\leftarrow})} := \sum_{i \geq 1} \delta_{(t\mathfrak{U}_i, \mathfrak{U}'_i, \overline{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))} \\ &\xrightarrow{m} \mathbb{V}_t^{m \circ (I, I, \overline{\Pi}^{\leftarrow})} := \sum_{i \geq 1} \delta_{m(t\mathfrak{U}_i, \mathfrak{U}'_i, \overline{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))}. \end{aligned}$$

Their trimmed counterparts are similarly defined by setting, for $r \in \mathbb{N}$,

$$\begin{aligned} {}^{(r)}\mathbb{V}'_t \xrightarrow{(I, I, \bar{\Pi}^{\leftarrow})} {}^{(r)}\mathbb{V}'_t^{(I, I, \bar{\Pi}^{\leftarrow})} &:= \sum_{i>r} \delta_{(t\mathfrak{U}_i, \mathfrak{U}'_i, \bar{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))} \\ \xrightarrow{m} {}^{(r)}\mathbb{V}'_t^{m \circ (I, I, \bar{\Pi}^{\leftarrow})} &:= \sum_{i>r} \delta_{m(t\mathfrak{U}_i, \mathfrak{U}'_i, \bar{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))}. \end{aligned}$$

When $\bar{\Pi}^+(0+) = \infty$ we can contrive $\bar{\Pi}^{+, \leftarrow}$ as a transformation of $(0, \infty)$ into $(0, \infty)$ and we will consider the image measures of \mathbb{V}_t and ${}^{(r)}\mathbb{V}_t$ under $(I, \bar{\Pi}^{+, \leftarrow}) : [0, t] \times (0, \infty) \rightarrow [0, \infty) \times (0, \infty)$, defined by

$$\mathbb{V}_t^{(I, \bar{\Pi}^{+, \leftarrow})} := \sum_{i \geq 1} \delta_{(t\mathfrak{U}_i, \bar{\Pi}^{+, \leftarrow}(\mathfrak{S}_i/t))} \quad \text{and} \quad {}^{(r)}\mathbb{V}_t^{(I, \bar{\Pi}^{+, \leftarrow})} := \sum_{i>r} \delta_{(t\mathfrak{U}_i, \bar{\Pi}^{+, \leftarrow}(\mathfrak{S}_i/t))}.$$

2.3. Representations for r -trimmed PPPs

In this section, the original point process \mathbb{X} , its ordered jumps, and the trimmed point process, is related to a corresponding standard version \mathbb{V} .

Lemma 1. *Let $t > 0$ and $r \in \mathbb{N}$.*

(i) *If $\bar{\Pi}(0+) = \infty$, we have the following distributional equivalences:*

$$\mathbb{X}_t \stackrel{D}{=} \mathbb{V}'_t^{m \circ (I, I, \bar{\Pi}^{\leftarrow})}, \quad (2.16)$$

$$(\tilde{T}^{(i)}(\mathbb{X}_t), \widetilde{\Delta X}_t^{(i)})_{i \geq 1} \stackrel{D}{=} (m(t\mathfrak{U}_i, \mathfrak{U}'_i, \bar{\Pi}^{\leftarrow}(\mathfrak{S}_i/t)))_{i \geq 1}, \quad (2.17)$$

$$\begin{aligned} \{(\tilde{T}^{(i)}(\mathbb{X}_t), \widetilde{\Delta X}_t^{(i)})_{1 \leq i \leq r}, {}^{(r)}\tilde{\mathbb{X}}_t\} \\ \stackrel{D}{=} \{(m(t\mathfrak{U}_i, \mathfrak{U}'_i, \bar{\Pi}^{\leftarrow}(\mathfrak{S}_i/t)))_{1 \leq i \leq r}, {}^{(r)}\mathbb{V}'_t^{m \circ (I, I, \bar{\Pi}^{\leftarrow})}\}. \end{aligned} \quad (2.18)$$

(ii) *If $\bar{\Pi}^+(0+) = \infty$, we have the following distributional equivalences:*

$$\begin{aligned} \mathbb{X}_t^+ &\stackrel{D}{=} \mathbb{V}_t^{(I, \bar{\Pi}^{+, \leftarrow})}, \\ (T^{(i)}(\mathbb{X}_t^+), \Delta X_t^{(i)})_{i \geq 1} &\stackrel{D}{=} (t\mathfrak{U}_i, \bar{\Pi}^{+, \leftarrow}(\mathfrak{S}_i/t))_{i \geq 1}, \\ \{(T^{(i)}(\mathbb{X}_t^+), \Delta X_t^{(i)})_{1 \leq i \leq r}, {}^{(r)}\mathbb{X}_t^+\} &\stackrel{D}{=} \{(t\mathfrak{U}_i, \bar{\Pi}^{+, \leftarrow}(\mathfrak{S}_i/t))_{1 \leq i \leq r}, {}^{(r)}\mathbb{V}_t^{(I, \bar{\Pi}^{+, \leftarrow})}\}. \end{aligned}$$

Proof. (i) Assume $\bar{\Pi}(0+) = \infty$, and introduce

$$\tilde{m} : (0, 1) \times (0, \infty) \rightarrow \mathbb{R}_*, \quad \tilde{m}(u', x) := x \mathbf{1}_{u' < g^+(x)} - x \mathbf{1}_{u' \geq g^+(x)}.$$

(The mapping \tilde{m} is the same as the m in (2.15) without the time component.)

Let $\mu^T := \mu \circ T^{-1}$ denote the image measure of a measure μ under a transformation T . Using this notation, and in view of (2.14), we get from $(dv)^{\bar{\Pi}^{\leftarrow}} = d\Pi^{|\cdot|}$ that

$$\begin{aligned} (du' \otimes dv)^{\tilde{m} \circ (I, \bar{\Pi}^{\leftarrow})}((x, \infty)) &= (du' \otimes d\Pi^{|\cdot|})(\tilde{m}^{-1}((x, \infty))) \\ &= \int_{(x, \infty)} g^+(v) \Pi^{|\cdot|}(dv) \\ &= \bar{\Pi}^+(x), \quad x > 0, \end{aligned} \tag{2.19}$$

and similarly with (x, ∞) replaced by $(-\infty, -x)$, and g^+ , $\bar{\Pi}^+$, replaced by g^- , $\bar{\Pi}^-$. With m as in (2.15), and since the tail functions determine the corresponding measures, (2.19) extends to

$$(ds \otimes du' \otimes dv)^{m \circ (I, \bar{\Pi}^{\leftarrow})} = ds \otimes d\Pi. \tag{2.20}$$

Let $h := m \circ (I, \bar{\Pi}^{\leftarrow})$. It follows from (2.20) that \mathbb{X}_t and $\mathbb{V}_t^{m \circ (I, \bar{\Pi}^{\leftarrow})} = \mathbb{V}_t^h$ share a common intensity measure $ds \otimes d\Pi$. Since both \mathbb{X} and \mathbb{V}^h are simple PPPs, this completes the proof of (2.16).

In order to show (2.17), introduce record times, defined recursively by

$$R_n := \min\{i > R_{n-1} : \bar{\Pi}^{\leftarrow}(\mathfrak{S}_i/t) > \bar{\Pi}^{\leftarrow}(\mathfrak{S}_{R_{n-1}}/t)\}, \quad R_1 := 1, n = 2, 3, 4, \dots$$

Observe that $(R_n)_{n \geq 1}$ is independent of (\mathfrak{U}_i) and (\mathfrak{U}'_i) .

Construct the sequence $(\tilde{T}^{(i)}(\mathbb{V}_t^h), \widetilde{\Delta X}^{(i)}(\mathbb{V}_t^h))_{i \geq 1}$ associated with trimming the process \mathbb{V}_t^h by choosing a sequence of independent permutations $(\sigma_n)_{n \geq 1}$, where

$$\sigma_n : \{R_{n-1}, \dots, R_n - 1\} \xrightarrow{1:1} \{R_{n-1}, \dots, R_n - 1\}, \quad n = 2, 3, 4, \dots,$$

are chosen according to the discrete uniform distribution amongst the finitely many candidates, independently of $(X_t)_{t \geq 0}$. By our construction of trimming, the pairs $(\{R_n\}, \{\sigma_n\})$ and $(\{\mathfrak{U}_i\}, \{\mathfrak{U}'_i\})$ are also independent. Consequently,

$$\begin{aligned} &\{\tilde{T}^{(i)}(\mathbb{V}_t^h), \widetilde{\Delta X}^{(i)}(\mathbb{V}_t^h)\}_{i \geq 1} \\ &= \{(\tilde{T}^{(i)}(\mathbb{V}_t^h), \widetilde{\Delta X}^{(i)}(\mathbb{V}_t^h))_{R_{n-1} \leq i < R_n}\}_{n \geq 2} \\ &= \{m(t\mathfrak{U}_{\sigma_n(i)}, \mathfrak{U}'_{\sigma_n(i)}, \bar{\Pi}^{\leftarrow}(\mathfrak{S}_{R_{n-1}}/t))_{R_{n-1} \leq i < R_n}\}_{n \geq 2} \\ &\stackrel{D}{=} \{m(t\mathfrak{U}_i, \mathfrak{U}'_i, \bar{\Pi}^{\leftarrow}(\mathfrak{S}_{R_{n-1}}/t))_{R_{n-1} \leq i < R_n}\}_{n \geq 2} \\ &= \{m(t\mathfrak{U}_i, \mathfrak{U}'_i, \bar{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))\}_{i \geq 1}. \end{aligned}$$

In view of (2.16), this completes the proof of (2.17). Note that (2.18) follows from (2.17). Part (ii) is shown analogously. \square

Next is our main theorem giving the representation for trimmed PPPs. For $x > 0$, write $\mathbb{X}_t^{+\cdot < x}$ and $\mathbb{X}_t^{|\cdot| < x}$ for point processes generated by deleting all points in \mathbb{X}_t^+ and \mathbb{X}_t not lying in the regions $[0, t] \times (0, x)$ and $[0, t] \times (-x, x)_*$, respectively:

$$\mathbb{X}_t^{+\cdot < x}(\cdot) := \mathbb{X}^+([0, t] \times (0, x) \cap \cdot)$$

and

$$\mathbb{X}_t^{|\cdot| < x}(\cdot) := \mathbb{X}([0, t] \times (-x, x)_* \cap \cdot).$$

Theorem 1. *Assume that \mathbb{X} , (\mathfrak{U}_i) , (\mathfrak{U}'_i) , \mathfrak{S}_r , $Y^\pm = (Y^\pm(t))_{t \geq 0}$, $Y = (Y(t))_{t \geq 0}$, are independent processes, with Y^\pm and Y being standard Poisson processes.*

(i) *Assume $\bar{\Pi}(0+) = \infty$. Then, for all $t > 0$, $r \in \mathbb{N}$,*

$$\begin{aligned} & (|\widetilde{\Delta X}_t^{(r)}|, {}^{(r)}\widetilde{\mathbb{X}}_t) \\ & \stackrel{\text{D}}{=} \left(\bar{\Pi}^{\leftarrow}(v), \mathbb{X}_t^{|\cdot| < \bar{\Pi}^{\leftarrow}(v)} + \sum_{i=1}^{Y^+(t\kappa^+(v))} \delta_{(t\mathfrak{U}_i, \bar{\Pi}^{\leftarrow}(v))} + \sum_{i=1}^{Y^-(t\kappa^-(v))} \delta_{(t\mathfrak{U}'_i, -\bar{\Pi}^{\leftarrow}(v))} \right)_{v=\mathfrak{S}_r/t}, \end{aligned} \quad (2.21)$$

where $\kappa^\pm(v)$ are the quantities in (2.7).

(ii) *Assume $\bar{\Pi}^+(0+) = \infty$. Then for all $t > 0$, $r \in \mathbb{N}$,*

$$(\Delta X_t^{(r)}, {}^{(r)}\mathbb{X}_t^+) \stackrel{\text{D}}{=} \left(\bar{\Pi}^{+, \leftarrow}(v), \mathbb{X}_t^{+\cdot < \bar{\Pi}^{+, \leftarrow}(v)} + \sum_{i=1}^{Y(t\kappa(v))} \delta_{(t\mathfrak{U}_i, \bar{\Pi}^{+, \leftarrow}(v))} \right)_{v=\mathfrak{S}_r/t},$$

where $\kappa(v) = \bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v) -) - v$.

Proof. Let $t > 0$, $r \in \mathbb{N}$, and introduce a point measure $\widetilde{\mathbb{V}}'_t$ as follows:

$$\widetilde{\mathbb{V}}'_t := \sum_{i \geq 1} \delta_{(t\mathfrak{U}_{i+r}, \mathfrak{U}'_{i+r}, (\mathfrak{S}_{i+r} - \mathfrak{S}_r)/t)}.$$

Then $\widetilde{\mathbb{V}}'_t$ is independent of $\mathfrak{V} := \mathfrak{S}_r/t$ with $\widetilde{\mathbb{V}}'_t \stackrel{\text{D}}{=} \mathbb{V}'_t$. Observe that

$$\begin{aligned} & E \exp \left\{ -\lambda \mathfrak{V} - \int f \, d\{\delta_{(0,0,\mathfrak{V})} \star \widetilde{\mathbb{V}}'_t\} \right\} \\ & = E \exp \left\{ -\lambda \mathfrak{V} - \int_0^t \int_0^1 \int_{\mathfrak{V}}^\infty (1 - e^{-f(s,u',v)}) \, ds \, du' \, dv \right\} \\ & = E \exp \left\{ -\lambda \mathfrak{V} - \int f \, d\widetilde{\mathbb{V}}'_t \right\}, \end{aligned} \quad (2.22)$$

for all nonnegative Borelian f and $\lambda \geq 0$. Here $\tilde{\mathbb{V}}'_t{}^{\geq v}(\cdot) := \tilde{\mathbb{V}}'_t([0, t] \times (0, 1) \times [v, \infty) \cap \cdot)$.

Assume $\bar{\Pi}(0+) = \infty$. Combining (2.18) and (2.22) yields

$$\begin{aligned} (|\widehat{\Delta X}_t^{(r)}|, {}^{(r)}\tilde{\mathbb{X}}_t) &\stackrel{\text{D}}{=} (\bar{\Pi}^{\leftarrow}(\mathfrak{V}), \{\delta_{(0,0,\mathfrak{V})} \star \tilde{\mathbb{V}}'_t\}^{m \circ (I, \bar{\Pi}^{\leftarrow})}) \\ &\stackrel{\text{D}}{=} (\bar{\Pi}^{\leftarrow}(\mathfrak{V}), \{\tilde{\mathbb{V}}'_t{}^{\geq \mathfrak{V}}\}^{m \circ (I, \bar{\Pi}^{\leftarrow})}). \end{aligned} \quad (2.23)$$

Next, set $\mathbb{Y}_t := \{\tilde{\mathbb{V}}'_t{}^{\geq \mathfrak{V}}\}^{m \circ (I, \bar{\Pi}^{\leftarrow})}$, and let $\mathbb{Y}_t^{|\cdot| < y}$, $\mathbb{Y}_t^{\geq y}$, $\mathbb{Y}_t^{\leq y}$ be the point processes obtained from \mathbb{Y}_t by removal of points not lying in the regions $[0, t] \times (-y, y)_*$, $[0, t] \times [y, \infty)$, $[0, t] \times (-\infty, y]$, respectively.

Let $\lambda \geq 0$ and $f_{-1}, f_0, f_1 : [0, t] \times \mathbb{R}_* \rightarrow [0, \infty]$ be Borel functions. Define $\Phi : [0, t] \times \mathbb{R}_* \times (0, \infty) \rightarrow [0, \infty]$ by setting

$$\Phi[s, x, y] := f_0(s, x) \mathbf{1}_{(0, y)}(|x|) + f_1(s, x) \mathbf{1}_{[y, \infty)}(x) + f_{-1}(s, x) \mathbf{1}_{(-\infty, -y]}(x).$$

Observe that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}_*} (f_0 d\mathbb{Y}_t^{|\cdot| < \bar{\Pi}^{\leftarrow}(\mathfrak{V})} + f_1 d\mathbb{Y}_t^{\geq \bar{\Pi}^{\leftarrow}(\mathfrak{V})} + f_{-1} d\mathbb{Y}_t^{\leq -\bar{\Pi}^{\leftarrow}(\mathfrak{V})}) \\ &= \int_0^t \int_{\mathbb{R}_*} \Phi[\cdot, \cdot, \bar{\Pi}^{\leftarrow}(\mathfrak{V})] d\mathbb{Y}_t \end{aligned} \quad (2.24)$$

and

$$\begin{aligned} &E \exp \left\{ -\lambda \mathfrak{V} - \int_0^t \int_{\mathbb{R}_*} \Phi[s, x, \bar{\Pi}^{\leftarrow}(\mathfrak{V})] \mathbb{Y}_t(ds, dx) \right\} \\ &= E \exp \left\{ -\lambda \mathfrak{V} - \int_0^t \int_0^1 \int_{\mathfrak{V}} \Phi[m(s, u', \bar{\Pi}^{\leftarrow}(v)), \bar{\Pi}^{\leftarrow}(\mathfrak{V})] \tilde{\mathbb{V}}'_t(ds, du', dv) \right\} \\ &= E \exp \left\{ -\lambda \mathfrak{V} - \int_0^t \int_0^1 \int_{\mathfrak{V}} (1 - \exp\{-\Phi[m(s, u', \bar{\Pi}^{\leftarrow}(v)), \bar{\Pi}^{\leftarrow}(\mathfrak{V})]\}) ds du' dv \right\}. \end{aligned} \quad (2.25)$$

As $\{v > 0 : \bar{\Pi}^{\leftarrow}(v) < \bar{\Pi}^{\leftarrow}(\mathfrak{V})\} \subseteq (\mathfrak{V}, \infty)$ and $\bar{\Pi}^{\leftarrow}(v) = \bar{\Pi}^{\leftarrow}(\mathfrak{V})$ for $v \in [\mathfrak{V}, \bar{\Pi}(\bar{\Pi}^{\leftarrow}(\mathfrak{V})-)]$, the last integral in the exponent equals

$$\begin{aligned} &\int_0^t \int_0^1 \int_0^\infty \mathbf{1}_{(0, \bar{\Pi}^{\leftarrow}(\mathfrak{V}))}(\bar{\Pi}^{\leftarrow}(v)) (1 - e^{-f_0(m(s, u', \bar{\Pi}^{\leftarrow}(v)))}) ds du' dv \\ &+ \kappa^+(\mathfrak{V}) \int_0^t (1 - e^{-f_1(s, \bar{\Pi}^{\leftarrow}(\mathfrak{V}))}) ds + \kappa^-(\mathfrak{V}) \int_0^t (1 - e^{-f_{-1}(s, -\bar{\Pi}^{\leftarrow}(\mathfrak{V}))}) ds, \end{aligned} \quad (2.26)$$

with $\kappa^\pm(v)$ as in (2.7). It follows from (2.20) and a change of variables that

$$\int_0^t \int_0^1 \int_0^\infty \mathbf{1}_{(0, \bar{\Pi}^{\leftarrow}(\mathfrak{V}))}(\bar{\Pi}^{\leftarrow}(v)) (1 - e^{-f_0(m(s, u', \bar{\Pi}^{\leftarrow}(v)))}) ds du' dv$$

$$= \int_0^t \int_{(-\bar{\Pi}^{\leftarrow}(\mathfrak{V}), \bar{\Pi}^{\leftarrow}(\mathfrak{V}))} (1 - e^{-f_0(s,x)}) ds \Pi(dx).$$

We get from (2.24), (2.25) and (2.26)

$$\begin{aligned} & E \exp \left\{ -\lambda \mathfrak{V} - \int_0^t \int_{\mathbb{R}_*} (f_0 d\mathbb{Y}_t^{|\cdot| < \bar{\Pi}^{\leftarrow}(\mathfrak{V})} + f_1 d\mathbb{Y}_t^{|\cdot| \geq \bar{\Pi}^{\leftarrow}(\mathfrak{V})} + f_{-1} d\mathbb{Y}_t^{|\cdot| \leq -\bar{\Pi}^{\leftarrow}(\mathfrak{V})}) \right\} \\ &= E \exp \left\{ -\lambda \mathfrak{V} - \mathbb{X}_t^{|\cdot| < \bar{\Pi}^{\leftarrow}(\mathfrak{V})}(f_0) \right. \\ &\quad \left. - \sum_{i=1}^{Y^+(t\kappa^+(\mathfrak{V}))} f_1(t\mathfrak{U}_i, \bar{\Pi}^{\leftarrow}(\mathfrak{V})) - \sum_{i=1}^{Y^-(t\kappa^-(\mathfrak{V}))} f_{-1}(t\mathfrak{U}'_i, -\bar{\Pi}^{\leftarrow}(\mathfrak{V})) \right\}, \end{aligned} \quad (2.27)$$

completing the proof of the following identity in law:

$$\begin{aligned} & (\mathfrak{V}, \mathbb{Y}_t^{|\cdot| < \bar{\Pi}^{\leftarrow}(\mathfrak{V})}, \mathbb{Y}_t^{|\cdot| \geq \bar{\Pi}^{\leftarrow}(\mathfrak{V})}, \mathbb{Y}_t^{|\cdot| \leq -\bar{\Pi}^{\leftarrow}(\mathfrak{V})}) \\ & \stackrel{D}{=} \left(\mathfrak{V}, \mathbb{X}_t^{|\cdot| < \bar{\Pi}^{\leftarrow}(\mathfrak{V})}, \sum_{i=1}^{Y^+(t\kappa^+(\mathfrak{V}))} \delta_{(t\mathfrak{U}_i, \bar{\Pi}^{\leftarrow}(\mathfrak{V}))}, \sum_{i=1}^{Y^-(t\kappa^-(\mathfrak{V}))} \delta_{(t\mathfrak{U}'_i, -\bar{\Pi}^{\leftarrow}(\mathfrak{V}))} \right), \end{aligned}$$

where $\mathfrak{V}, \mathbb{X}_t', Y^+, Y^-, (\mathfrak{U}_i), (\mathfrak{U}'_i)$ are independent with $\mathbb{X}_t' \stackrel{D}{=} \mathbb{X}_t$. The proof of part (i) is completed by combining (2.23) and (2.27). The proof of part (ii) is similar. \square

2.4. Representations for the r -trimmed Lévy processes

By the Lévy–Itô decomposition (Sato [47], Theorem 19.2, page 120), we can decompose a real-valued Lévy process X_t , defined on the probability space (Ω, \mathcal{F}, P) , as

$$X_t = \gamma t + \sigma Z_t + X_t^{(J)}, \quad t \geq 0, \quad (2.28)$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, $(Z_t)_{t \geq 0}$ is a standard Brownian motion, and $(X_t^{(J)})_{t \geq 0}$, the jump process of X , is independent of $(Z_t)_{t \geq 0}$. It satisfies, locally uniform in $t \geq 0$,

$$X_t^{(J)} = \text{a.s.} \lim_{\varepsilon \downarrow 0} \left(\sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > \varepsilon\}} - t \int_{\varepsilon < |x| \leq 1} x \Pi(dx) \right). \quad (2.29)$$

Now we can complete the proof of Theorem 2.1.

Proof of Theorem 2.1. We will prove part (i), the identity for the r -fold modulus trimmed Lévy process. Trimming of positive jumps as in part (ii) follows similarly. Let $t > 0$, $r \in \mathbb{N}$ be fixed. By (2.28) and the definition of ${}^{(r)}\tilde{X}_t$, the r -fold modulus trimmed

Lévy process is

$${}^{(r)}\widetilde{X}_t = \gamma t + \sigma Z_t + X_t^{(J)} - \sum_{i=1}^r \widetilde{\Delta X}_t^{(i)}, \quad t > 0.$$

Note that the jump process of ${}^{(r)}\widetilde{X}_t$ and its quadratic variation are obtained by applying the summing functional to the r -fold modulus trimmed point process ${}^{(r)}\widetilde{\mathbb{X}}$ and to the squared jumps of ${}^{(r)}\widetilde{\mathbb{X}}$. Using (2.29), we can write

$$X_t^{(J)} - \sum_{i=1}^r \widetilde{\Delta X}_t^{(i)} = \text{a.s.} \lim_{\varepsilon \downarrow 0} \left(\int_{[0,t] \times \{|x| > \varepsilon\}} x {}^{(r)}\widetilde{\mathbb{X}}(\mathrm{d}s, \mathrm{d}x) - t \int_{\varepsilon < |x| \leq 1} x \Pi(\mathrm{d}x) \right) \quad (2.30)$$

The corresponding r -trimmed quadratic variation is simply

$${}^{(r)}\widetilde{V}_t = \int_{[0,t] \times \mathbb{R}_*} x^2 {}^{(r)}\widetilde{\mathbb{X}}(\mathrm{d}s, \mathrm{d}x).$$

Recall from Lemma 1 and Theorem 1 that the distribution of ${}^{(r)}\widetilde{\mathbb{X}}_t$ can be decomposed as the superposition of three independent point measures, as in (2.21). Splitting the integral in (2.30) into these components gives

$$\begin{aligned} & \text{a.s.} \lim_{\varepsilon \downarrow 0} \left(\int_{[0,t] \times \{|x| > \varepsilon\}} x {}^{(r)}\widetilde{\mathbb{X}}(\mathrm{d}s, \mathrm{d}x) - t \int_{\varepsilon < |x| \leq 1} x \Pi(\mathrm{d}x) \right) \\ & \stackrel{\text{D}}{=} \text{a.s.} \lim_{\varepsilon \downarrow 0} \left(\int_{[0,t] \times \{|x| > \varepsilon\}} x \mathbb{X}^{|\cdot| < \overline{\Pi}^{\leftarrow}(\mathfrak{S}_r/t)}(\mathrm{d}s, \mathrm{d}x) - t \int_{\varepsilon < |x| \leq 1} x \Pi(\mathrm{d}x) \right) \\ & \quad + \overline{\Pi}^{\leftarrow}(\mathfrak{S}_r/t)(Y^+(t\kappa^+(\mathfrak{S}_r/t)) - Y^-(t\kappa^-(\mathfrak{S}_r/t))). \end{aligned}$$

A similar expression holds for ${}^{(r)}\widetilde{V}_t$. Thus, we conclude

$$\begin{aligned} ({}^{(r)}\widetilde{X}_t, {}^{(r)}\widetilde{V}_t, |\widetilde{\Delta X}_t^{(r)}|) & \stackrel{\text{D}}{=} \{\widetilde{X}_t^v + \overline{\Pi}^{\leftarrow}(v)(Y_{t\kappa^+(v)}^+ - Y_{t\kappa^-(v)}^-), \\ & \quad \widetilde{V}_t^v + \overline{\Pi}^{\leftarrow}(v)^2(Y_{t\kappa^+(v)}^+ + Y_{t\kappa^-(v)}^-), \overline{\Pi}^{\leftarrow}(v)\}_{v=\mathfrak{S}_r/t}. \end{aligned}$$

This is (2.9) and completes the proof of part (i). \square

This completes our derivation of the trimming identities. In the next sections, we turn to applications of them.

3. X comparable with its large jump processes

In this section, we apply Theorem 2.1 to complete a result of Maller and Mason [38] concerning the ratio of the process to its jump of largest magnitude. Note that when

$\overline{\Pi}(0+) = \infty$, we have $|\widetilde{\Delta X}_t^{(1)}| = \sup_{0 < s \leq t} |\Delta X_s| > 0$ a.s. for all $t > 0$; similarly, when $\overline{\Pi}^+(0+) = \infty$, $\Delta X_t^{(1)} = \sup_{0 < s \leq t} \Delta X_s > 0$ a.s. for all $t > 0$. Recall that $\overline{\Pi}(x)$ is said to be slowly varying (SV) as $x \downarrow 0$ if $\lim_{x \downarrow 0} \overline{\Pi}(ux)/\overline{\Pi}(x) = 1$ for all $u > 0$ (e.g., Bingham, Goldie and Teugels [7]).

Theorem 2. *Suppose $\sigma^2 = 0$ and $\overline{\Pi}(0+) = \infty$. Then*

$$\frac{X_t}{\widetilde{\Delta X}_t^{(1)}} \xrightarrow{P} 1, \quad \text{as } t \downarrow 0, \quad (3.1)$$

iff $\overline{\Pi}(x) \in \text{SV}$ at 0 (so that X is of bounded variation) and X has drift 0. These imply

$$\frac{|\widetilde{\Delta X}_t^{(2)}|}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0; \quad (3.2)$$

and conversely (3.2) implies $\overline{\Pi}(x) \in \text{SV}$ at 0.

For the proof, we need two preliminary lemmas. The first calculates a distribution related to the large jumps, and the second applies Theorem 2.1 to derive a useful inequality.

Lemma 2. *Assume $\overline{\Pi}(0+) = \infty$. Then for $t > 0$, $0 < u < 1$,*

$$P(|\widetilde{\Delta X}_t^{(2)}| \leq u |\widetilde{\Delta X}_t^{(1)}|) = t \int_{(0, \infty)} e^{-t\overline{\Pi}(u\overline{\Pi}^{\leftarrow}(v))} dv. \quad (3.3)$$

A similar expression to (3.3) is true when $\overline{\Pi}^+(0+) = \infty$, with $|\widetilde{\Delta X}_t^{(1)}|$ and $|\widetilde{\Delta X}_t^{(2)}|$ replaced by $\Delta X_t^{(1)}$ and $\Delta X_t^{(2)}$, and $\overline{\Pi}$ and $\overline{\Pi}^{\leftarrow}$ replaced by $\overline{\Pi}^+$ and $\overline{\Pi}^{+, \leftarrow}$.

Proof. Assume $\overline{\Pi}(0+) = \infty$ and take $t > 0$. We get from (2.17) that

$$(|\widetilde{\Delta X}_t^{(1)}|, |\widetilde{\Delta X}_t^{(2)}|) \stackrel{D}{=} (\overline{\Pi}^{\leftarrow}(\mathfrak{E}_1/t), \overline{\Pi}^{\leftarrow}((\mathfrak{E}_1 + \mathfrak{E}_2)/t)), \quad (3.4)$$

where \mathfrak{E}_1 and \mathfrak{E}_2 are independent unit exponential random variables. Take $0 < u < 1$ and $v > 0$ and let $y_{t,u}(v) := t\overline{\Pi}(u\overline{\Pi}^{\leftarrow}(v/t))$. Then, in view of (3.4),

$$\begin{aligned} P(|\widetilde{\Delta X}_t^{(2)}| \leq u |\widetilde{\Delta X}_t^{(1)}|) &= P(\overline{\Pi}^{\leftarrow}((\mathfrak{E}_1 + \mathfrak{E}_2)/t) \leq u \overline{\Pi}^{\leftarrow}(\mathfrak{E}_1/t)) \\ &= P(\mathfrak{E}_1 + \mathfrak{E}_2 \geq y_{t,u}(\mathfrak{E}_1)) \\ &= \int_{(0, \infty)} e^{-(y_{t,u}(v)-v)} e^{-v} dv \\ &= \int_{(0, \infty)} \exp\{-t\overline{\Pi}(u\overline{\Pi}^{\leftarrow}(v/t))\} dv. \end{aligned}$$

Changing the variable from v/t to v gives (3.3). The version for large jumps, rather than jumps large in modulus, is proved similarly. \square

Lemma 3. Assume $\bar{\Pi}(0+) = \infty$, and let a_t be any nonstochastic function in \mathbb{R} . Then for $t > 0$ and $0 < u < 1/4$,

$$4P(|^{(1)}\tilde{X}_t - a_t| > u|\widetilde{\Delta X}_t^{(1)}|) \geq P(|\widetilde{\Delta X}_t^{(2)}| > 4u|\widetilde{\Delta X}_t^{(1)}|). \quad (3.5)$$

Assuming $\bar{\Pi}^+(0+) = \infty$, the same inequality (3.5) holds with $^{(1)}X_t$, $\Delta X_t^{(1)}$ and $\Delta X_t^{(2)}$ in place of $^{(1)}\tilde{X}_t$, $|\widetilde{\Delta X}_t^{(1)}|$ and $|\widetilde{\Delta X}_t^{(2)}|$.

Proof. Let \mathfrak{E} be an exponential random variable with $E\mathfrak{E} = 1$, thus, $\mathfrak{E} \stackrel{D}{=} \mathfrak{S}_1$. Using the identity in (2.12) with $r = 1$, the left-hand side of (3.5) is, for $u > 0$,

$$\int_0^\infty 4P(|\tilde{X}_t^v + \tilde{G}_t^v - a_t| > uy_v)P(\mathfrak{E} \in t dv), \quad (3.6)$$

where we abbreviate $y_v := \bar{\Pi}^{\leftarrow}(v)$, $v > 0$. For each $v > 0$, let $(\bar{X}_t^v)_{t \geq 0}$ and $(\bar{G}_t^v)_{t \geq 0}$ be independent copies of $(\tilde{X}_t^v)_{t \geq 0}$ and $(\tilde{G}_t^v)_{t \geq 0}$, with $(\bar{G}_t^v)_{t \geq 0}$ also independent of $(\bar{X}_t^v)_{t \geq 0}$. Define the symmetrised process $(\hat{Y}_t^v)_{t \geq 0}$ by

$$\hat{Y}_t^v = (\tilde{X}_t^v + \tilde{G}_t^v) - (\bar{X}_t^v + \bar{G}_t^v), \quad t > 0,$$

with jump process $\Delta \hat{Y}_t^v = \hat{Y}_t^v - \hat{Y}_{t-}^v$, $t > 0$. Then the integrand in (3.6) satisfies

$$\begin{aligned} 4P(|\tilde{X}_t^v + \tilde{G}_t^v - a_t| > uy_v) &= 2P(|\tilde{X}_t^v + \tilde{G}_t^v - a_t| > uy_v) + 2P(|\bar{X}_t^v + \bar{G}_t^v - a_t| > uy_v) \\ &\geq 2P(|(\tilde{X}_t^v + \tilde{G}_t^v - a_t) - (\bar{X}_t^v + \bar{G}_t^v - a_t)| > 2uy_v) \\ &= 2P(|\hat{Y}_t^v| > 2uy_v). \end{aligned} \quad (3.7)$$

Substitute the inequality (3.7) in (3.6) and equate to the left-hand side of (3.5) to get

$$4P(|^{(1)}\tilde{X}_t - a_t| > u|\widetilde{\Delta X}_t^{(1)}|) \geq 2 \int_0^\infty P(|\hat{Y}_t^v| > 2uy_v)P(\mathfrak{E} \in t dv), \quad u > 0. \quad (3.8)$$

Take $u \in (0, 1/4)$. Applying Lévy's maximal inequality for random walks (Feller ([17], Lemma 2, page 147), we have

$$\begin{aligned} 2P(|\hat{Y}_t^v| > 2uy_v) &= 2 \lim_{n \rightarrow \infty} P\left(\left|\sum_{i=1}^{2^n} (\hat{Y}_{it/2^n}^v - \hat{Y}_{(i-1)t/2^n}^v)\right| > 2uy_v\right) \\ &\geq \lim_{n \rightarrow \infty} P\left(\max_{1 \leq j \leq 2^n} |\hat{Y}_{jt/2^n}^v - \hat{Y}_{(j-1)t/2^n}^v| > 2uy_v\right) \\ &\geq P\left(\sup_{0 < s \leq t} |\Delta \hat{Y}_s^v| > 4uy_v\right). \end{aligned} \quad (3.9)$$

The Lévy measure of \tilde{X}_t^v is $\Pi(dx)\mathbf{1}_{\{|x|<y_v\}}$, $x \in \mathbb{R}$, having tail function

$$(\bar{\Pi}(x) - \bar{\Pi}(y_v-))\mathbf{1}_{\{x < y_v\}}, \quad x > 0.$$

Suppose at first that $\Delta\bar{\Pi}(y_v) > 0$. Then \tilde{G}_t^v is nonzero. Its Lévy measure consists of point masses at $\pm y_v$ with magnitudes $\kappa^\pm(v)$, given by (2.7). Hence, it has tail

$$(\bar{\Pi}(y_v-) - v) \frac{(\Delta\bar{\Pi}(y_v) + \Delta\bar{\Pi}(-y_v))}{\Delta\bar{\Pi}(y_v)} \mathbf{1}_{\{x < y_v\}} = (\bar{\Pi}(y_v-) - v) \mathbf{1}_{\{x < y_v\}}, \quad x > 0.$$

Adding the two tails gives the tail of $\tilde{X}_t^v + \tilde{G}_t^v$ as $(\bar{\Pi}(x) - v)\mathbf{1}_{\{x < y_v\}}$, $x > 0$. The symmetrisation \hat{Y}_t^v has Lévy tail being twice the magnitude of this. This result remains true when $\Delta\bar{\Pi}(y_v) = 0$, as $\tilde{G}_t^v \equiv 0$ and $\bar{\Pi}(y_v-) = v$ then.

We can now calculate the right-hand side of (3.9) and deduce from it that

$$\begin{aligned} 2P(|\hat{Y}_t^v| > 2uy_v) &\geq 1 - e^{-2t(\bar{\Pi}(4uy_v) - v)} \\ &\geq 1 - e^{-t(\bar{\Pi}(4uy_v) - v)}. \end{aligned} \quad (3.10)$$

Finally, (3.8), (3.10) and Lemma 2 give

$$\begin{aligned} 4P(|^{(1)}\tilde{X}_t - a_t| > u|\widetilde{\Delta X}_t^{(1)}|) &\geq t \int_0^\infty (e^{-tv} - e^{-t\bar{\Pi}(4uy_v)}) dv \\ &= P(|\widetilde{\Delta X}_t^{(2)}| > 4u|\widetilde{\Delta X}_t^{(1)}|). \end{aligned}$$

This proves (3.5). To derive the version for $^{(1)}X_t$, define the one-sided Lévy process X_t^* having triplet $(\gamma, \sigma^2, \Pi^*(dx) = \Pi(dx)\mathbf{1}_{\{x>0\}})$, and let $\widetilde{\Delta X}_t^{*,(r)}$ be the jump of r th largest modulus up until time t for $(X_t^*)_{t \geq 0}$, $r \in \mathbb{N}$. Then $\widetilde{\Delta X}_t^{*,(r)} = \Delta X_t^{(r)}$ and $^{(1)}X_t = ^{(1)}\tilde{X}_t^* = X_t^* - \widetilde{\Delta X}_t^{*,(1)}$. Assuming $\bar{\Pi}^+(0+) = \infty$, inequality (3.5) with $^{(1)}X_t$, $\Delta X_t^{(1)}$ and $\Delta X_t^{(2)}$ replacing $^{(1)}\tilde{X}_t$, $|\widetilde{\Delta X}_t^{(1)}|$ and $|\widetilde{\Delta X}_t^{(2)}|$ then follows from (3.5) itself, applied to X_t^* . \square

Lemma 4. Assume $\bar{\Pi}(0+) = \infty$. Then

$$\frac{|\widetilde{\Delta X}_t^{(2)}|}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0, \quad (3.11)$$

implies $\bar{\Pi}(x)$ is SV at 0.

Proof. From (3.3), for $0 < u < 1$, with $y_v := \bar{\Pi}^\leftarrow(v)$,

$$P(|\widetilde{\Delta X}_t^{(2)}| > u|\widetilde{\Delta X}_t^{(1)}|) = t \int_0^\infty (e^{-tv} - e^{-t\bar{\Pi}(uy_v)}) dv$$

$$= \int_0^\infty (e^{-v} - e^{-t\bar{\Pi}(uy_{v/t})}) dv. \quad (3.12)$$

Assume (3.11), so the integral on the right-hand side of (3.12) tends to 0 as $t \downarrow 0$. Take any sequence $t_k \downarrow 0$ and by Helly's theorem select for each $u > 0$ a subsequence $t_{k'} = t_{k'}(u) \downarrow 0$ such that $t_{k'}\bar{\Pi}(uy_{v/t_{k'}})$ converges vaguely to $g_u(v)$, as $k' \rightarrow \infty$, where $g_u(v)$ is a monotone function of v . Since $t\bar{\Pi}(uy_{v/t}) \geq t\bar{\Pi}(y_{v/t}) \geq v$, we have $g_u(v) \geq v$. Fatou's lemma applied to (3.12) shows then that $g_u(v) = v$ for $v > 0$, thus $t_{k'}\bar{\Pi}(uy_{v/t_{k'}}) \rightarrow v$, and since this is true for all subsequences we deduce

$$\lim_{t \downarrow 0} t\bar{\Pi}(uy_{v/t}) = v, \quad v > 0, 0 < u < 1.$$

Given $x > 0$, $v > 0$, let $t(x) = v/\bar{\Pi}(x)$. Then $y_{v/t(x)} = \bar{\Pi}^\leftarrow(\bar{\Pi}(x)) \leq x$, implying $\bar{\Pi}(uy_{v/t(x)}) \geq \bar{\Pi}(ux)$. So we get, for $0 < u < 1$,

$$1 \leq \frac{\bar{\Pi}(ux)}{\bar{\Pi}(x)} \leq \frac{t(x)\bar{\Pi}(uy_{v/t(x)})}{v} \rightarrow \frac{v}{v} = 1, \quad \text{as } x \downarrow 0,$$

and $\bar{\Pi} \in SV$ at 0. □

Proof of Theorem 2. Observe that (3.1) is equivalent to

$$\frac{|^{(1)}\widetilde{X}_t|}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0,$$

and this implies (3.11) by Lemma 3. Thus, by Lemma 4, $\bar{\Pi} \in SV$ at 0. Hence, $\int_0^1 \bar{\Pi}(x) dx < \infty$ and X is of bounded variation, with drift d_X . By, for example, Bertoin ([5], Proposition 11, page 167), $X_t/t \xrightarrow{P} d_X$ as $t \downarrow 0$, while, for any $\delta > 0$,

$$P\left(\sup_{0 < s \leq t} |\Delta X_s| > \delta t\right) = 1 - e^{-t\bar{\Pi}(\delta t)} \rightarrow 0,$$

thus $\widetilde{\Delta X}_t^{(1)}/t \xrightarrow{P} 0$ as $t \downarrow 0$. But

$$\frac{|X_t|}{t} = \frac{|X_t|}{|\widetilde{\Delta X}_t^{(1)}|} \cdot \frac{|\widetilde{\Delta X}_t^{(1)}|}{t} \xrightarrow{P} (1)(0) = 0,$$

showing that $d_X = 0$.

Conversely, (3.1) holds when $\bar{\Pi} \in SV$ at 0 and $d_X = 0$, as shown in Lemma 5.1 of Maller and Mason [38]. □

The next result follows by applying Theorem 2 to the Lévy process $(\sum_{0 < s \leq t} |\Delta X_s|)_{t > 0}$, when X_t is of bounded variation.

Corollary 2. Suppose $\sigma^2 = 0$ and $\bar{\Pi}(0+) = \infty$. X_t is of bounded variation and

$$\frac{\sum_{0 < s \leq t} |\Delta X_s|}{\sup_{0 < s \leq t} |\Delta X_s|} \xrightarrow{P} 1, \quad \text{as } t \downarrow 0,$$

iff $\bar{\Pi}(x) \in SV$ at 0 (so that X is of bounded variation) and X has drift 0.

Remark 1. As another corollary of Theorem 2, it is not hard to show that $\bar{\Pi}(x) \in SV$ at 0 implies $t\bar{\Pi}(|X_t|) \xrightarrow{D} \mathfrak{E}$ as $t \downarrow 0$. The variance gamma model, widely used in financial modelling, has Lévy measure whose tail is slowly varying at 0 (Madan and Seneta ([35], page 519)). Our results for such processes provide useful intuition and, more specifically, may be of immediate use in applications, such as for estimation of Π or simulation, and so forth.

The next theorem gives a one-sided version of Theorem 2. Condition (3.13) reflects a kind of dominance of the positive part of X over its negative part. We defer the proof of Theorem 3 to the following section, where we study such dominance ideas in detail.

Theorem 3. Suppose $\bar{\Pi}^+(0+) = \infty$. Then

$$\frac{X_t}{\Delta X_t^{(1)}} \xrightarrow{P} 1, \quad \text{as } t \downarrow 0 \tag{3.13}$$

iff $\bar{\Pi}^+(x) \in SV$ at 0, X is of bounded variation with drift 0, and $\lim_{x \downarrow 0} \bar{\Pi}^-(x)/\bar{\Pi}^+(x) = 0$.

4. Comparing positive and negative jumps

In this section, we are concerned with comparing magnitudes of positive and negative jumps of X , in various ways. Define $\Delta X_t^+ = \max(\Delta X_t, 0)$, $\Delta X_t^- = \max(-\Delta X_t, 0)$, and

$$(\Delta X^+)_t^{(1)} = \sup_{0 < s \leq t} \Delta X_s^+ \quad \text{and} \quad (\Delta X^-)_t^{(1)} = \sup_{0 < s \leq t} \Delta X_s^-, \quad t > 0.$$

In the Poisson point process of jumps $(\Delta X_t)_{t > 0}$, the numbers of jumps and their magnitudes in disjoint regions are independent. Thus, the positive and negative jump processes are independent. When the integrals are finite, define

$$A_{\pm}(x) := \int_0^x \bar{\Pi}^{\pm}(y) dy = x \int_0^1 \bar{\Pi}^{\pm}(xy) dy.$$

We obtain the following.

Theorem 4. Suppose $\bar{\Pi}^\pm(0+) = \infty$. For (4.1) assume $\sum_{0 < s \leq t} \Delta X_s^-$ is finite a.s., and for (4.2) assume $\sum_{0 < s \leq t} \Delta X_s^+$ is finite a.s. For (4.3), assume both are finite a.s. Then

$$\frac{\sum_{0 < s \leq t} \Delta X_s^-}{\sup_{0 < s \leq t} \Delta X_s^+} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0 \quad \text{if and only if} \quad \lim_{x \downarrow 0} \frac{\int_0^x \bar{\Pi}^-(y) dy}{x \bar{\Pi}^+(x)} = 0; \quad (4.1)$$

also

$$\frac{\sup_{0 < s \leq t} \Delta X_s^-}{\sum_{0 < s \leq t} \Delta X_s^+} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0 \quad \text{if and only if} \quad \lim_{x \downarrow 0} \frac{x \bar{\Pi}^-(x)}{\int_0^x \bar{\Pi}^+(y) dy} = 0; \quad (4.2)$$

and

$$\frac{\sum_{0 < s \leq t} \Delta X_s^-}{\sum_{0 < s \leq t} \Delta X_s^+} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0 \quad \text{if and only if} \quad \lim_{x \downarrow 0} \frac{\int_0^x \bar{\Pi}^-(y) dy}{\int_0^x \bar{\Pi}^+(y) dy} = 0. \quad (4.3)$$

Finally,

$$\frac{\sup_{0 < s \leq t} \Delta X_s^-}{\sup_{0 < s \leq t} \Delta X_s^+} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0 \quad \text{if and only if} \quad \lim_{x \downarrow 0} \frac{\bar{\Pi}^-(\varepsilon x)}{\bar{\Pi}^+(x)} = 0 \quad \text{for all } \varepsilon > 0. \quad (4.4)$$

Proof. To prove the equivalence in (4.1), note that, for any $\lambda > 0$,

$$\begin{aligned} E \exp \left(-\lambda \frac{\sum_{0 < s \leq t} \Delta X_s^-}{\sup_{0 < s \leq t} \Delta X_s^+} \right) &= EE \left[\exp \left(-\frac{\lambda}{(\Delta X^+)_t^{(1)}} \sum_{0 < s \leq t} \Delta X_s^- \right) \middle| (\Delta X^+)_t^{(1)} \right] \\ &= E \left[\exp \left(-t \int_{(0, \infty)} (1 - e^{-\lambda x / (\Delta X^+)_t^{(1)}}) \Pi^{(-)}(dx) \right) \right] \\ &= \int_{(0, \infty)} \exp \left(-t \int_{(0, \infty)} (1 - e^{-\lambda x / y}) \Pi^{(-)}(dx) \right) \lambda_t^+(dy), \end{aligned} \quad (4.5)$$

where

$$\lambda_t^+(x) := P \left(\sup_{0 < s \leq t} \Delta X_s \leq x \right) = e^{-t \bar{\Pi}^+(x)}, \quad x > 0, t > 0.$$

By (4.5), the left-hand relation in (4.1) holds if and only if, for all $\lambda > 0$,

$$\lim_{t \downarrow 0} \int_{(0, \infty)} (1 - e^{-t \int_{(0, \infty)} (1 - e^{-\lambda x / y}) \Pi^{(-)}(dx)}) \lambda_t^+(dy) = 0. \quad (4.6)$$

Use the lower bound in the inequalities (cf. Bertoin [5], Proposition 1, page 74)

$$(\lambda/3y) A_-(y/\lambda) \leq \int_{(0, \infty)} (1 - e^{-\lambda x / y}) \Pi^{(-)}(dx) \leq (\lambda/y) A_-(y/\lambda), \quad y > 0, \lambda > 0, \quad (4.7)$$

with $\lambda = 1$ to get a lower bound for the integral in (4.6) of

$$\int_{(0,\infty)} (1 - e^{-tA_-(y)/3y}) \lambda_t^+(dy) \geq \int_{(0,z]} (1 - e^{-tA_-(y)/3y}) \lambda_t^+(dy) \quad (4.8)$$

for any $z > 0$. It is easily checked that $A_-(z)/z$ is nonincreasing for $z > 0$, so the last integral in (4.8) is not smaller than

$$(1 - e^{-tA_-(z)/3z}) \lambda_t^+(z) = (1 - e^{-tA_-(z)/3z}) e^{-t\bar{\Pi}^+(z)}.$$

Now choose $t = 1/\bar{\Pi}^+(z)$ and let $t \downarrow 0$ (so $z \downarrow 0$) to get the righthand relation in (4.1).

Conversely, assume the right-hand relation in (4.1). Then the upper bound in (4.7) shows that the integral in (4.6) is no larger than

$$\int_{(0,\infty)} (1 - e^{-\lambda t A_-(y/\lambda)/y}) \lambda_t^+(dy).$$

This is a nondecreasing function of λ so it suffices to show that it tends to 0 as $t \downarrow 0$ for $\lambda > 1$. Then since A_- is nondecreasing, for any $z > 0$ the integral is bounded above by

$$\int_{[z,\infty)} (1 - e^{-\lambda t A_-(y)/y}) \lambda_t^+(dy) + \lambda_t^+(z-) \leq 1 - e^{-\lambda t A_-(z)/z} + e^{-t\bar{\Pi}^+(z-)}.$$

Take $t > 0$ and $a > 0$ and let $z = \bar{\Pi}^{+,\leftarrow}(a/t)$. Then the last expression is no larger than

$$1 - e^{-a\lambda A_-(z)/z\bar{\Pi}^+(z)} + e^{-a}.$$

Letting $t \downarrow 0$, so $z \downarrow 0$, then $a \rightarrow \infty$, this tends to 0 by the right-hand relation in (4.1).

The equivalence in (4.2) is proved similarly to that in (4.1), by reversing the numerator and denominator and interchanging $+/-$ and noting that the left-hand relation in (4.2) holds if and only if the Laplace transform of the ratio on the left of (4.2) tends to 1 as $t \downarrow 0$.

The equivalence in (4.3) can be inferred from that in (4.2) with the following device. The left-hand relation in (4.3) holds if and only if

$$\lim_{t \downarrow 0} E \exp \left(-\lambda \sum_{0 < s \leq t} \Delta X_s^- / \sum_{0 < s \leq t} \Delta X_s^+ \right) = 1 \quad \text{for all } \lambda > 0. \quad (4.9)$$

The Laplace transform on the left-hand side of (4.9) equals

$$\int_{(0,\infty)} \exp \left(-t \int_{(0,\infty)} (1 - e^{-x/y}) \Pi^{(-)}(dx) \right) P \left(\sum_{0 < s \leq t} \Delta X_s^+ \in \lambda dy \right). \quad (4.10)$$

Define a measure $\rho(\cdot)$ on $(0, \infty)$ in terms of its tail:

$$\bar{\rho}(y) = \int_{(0,\infty)} (1 - e^{-x/y}) \Pi^{(-)}(dx), \quad y > 0.$$

Then $\bar{\rho}(y)$ is nonincreasing, $\bar{\rho}(0+) = \infty$, $\bar{\rho}(+\infty) = 0$, and $\int_0^1 y\bar{\rho}(y) dy < \infty$. So ρ is a Lévy measure and we can define a Lévy process $(U_t)_{t \geq 0}$, independent of $(X_t)_{t \geq 0}$, having Lévy characteristics $(0, 0, \rho)$ and jump process $\Delta U_t := U_t - U_{t-}$, $t > 0$. Then

$$P\left(\sup_{0 < s \leq t} \Delta U_s \leq y\right) = e^{-t\bar{\rho}(y)}, \quad y > 0,$$

and the right-hand side of (4.10) is

$$\begin{aligned} \int_{(0, \infty)} e^{-t\bar{\rho}(y)} P\left(\sum_{0 < s \leq t} \Delta X_s^+ \in \lambda dy\right) &= \int_{(0, \infty)} P\left(\sup_{0 < s \leq t} \Delta U_s \leq y\right) P\left(\sum_{0 < s \leq t} \Delta X_s^+ \in \lambda dy\right) \\ &= P\left(\sup_{0 < s \leq t} \Delta U_s / \sum_{0 < s \leq t} \Delta X_s^+ \leq \lambda^{-1}\right). \end{aligned}$$

Thus (4.9) holds if and only if

$$\frac{\sup_{0 < s \leq t} \Delta U_s}{\sum_{0 < s \leq t} \Delta X_s^+} \xrightarrow{P} 0, \quad \text{as } t \downarrow 0.$$

Applying (4.2), with U_t in the role of the negative jump process, this is so if and only if

$$\lim_{y \downarrow 0} \frac{y\bar{\rho}(y)}{A_+(y)} = 0. \quad (4.11)$$

The estimates in (4.7) give

$$\frac{A_-(y)}{3y} \leq \bar{\rho}(y) \leq \frac{A_-(y)}{y},$$

so the equivalence in (4.3) follows from (4.11).

Finally, for the equivalence in (4.4), use

$$P\left(\sup_{0 < s \leq t} \Delta X_s^- > u \mid \sup_{0 < s \leq t} \Delta X_s^+\right) = \int_{(0, \infty)} (1 - e^{-t\bar{\Pi}^-(uy)}) \lambda_t^+(dy),$$

and similar calculations as above. \square

To complete this section, we give the deferred proof of Theorem 3.

Proof of Theorem 3. Assume $\bar{\Pi}^+(0+) = \infty$ and suppose first that (3.13) holds. Then the same proof as used for showing that (3.1) is equivalent to $\bar{\Pi}(x) \in SV$ at 0, shows here that $\bar{\Pi}^+(x) \in SV$ at 0. This implies that the Lévy process $\sum_{0 < s \leq t} \Delta X_s^+$ is of bounded variation, and so

$$\frac{\sum_{0 < s \leq t} \Delta X_s^+}{\sup_{0 < s \leq t} \Delta X_s^+} \xrightarrow{P} 1, \quad (4.12)$$

by Theorem 2 applied to $\sum_{0 < s \leq t} \Delta X_s^+$. Now, $\bar{\Pi}^+(x) \in SV$ at 0 implies $\int_0^1 \bar{\Pi}^+(y) dy < \infty$ hence $\lim_{x \downarrow 0} x \bar{\Pi}^+(x) = 0$. This means

$$P\left(\sup_{0 < s \leq t} \Delta X_s^+ > \delta t\right) \leq 1 - e^{-t \bar{\Pi}^+(\delta t)} \rightarrow 0, \quad \text{as } t \downarrow 0 \text{ for } \delta > 0,$$

thus $\sup_{0 < s \leq t} \Delta X_s^+ / t \xrightarrow{P} 0$. So by (3.13)

$$\frac{X_t}{t} = \frac{X_t}{\Delta X_t^{(1)}} \cdot \frac{\Delta X_t^{(1)}}{t} \xrightarrow{P} 0. \quad (4.13)$$

Then $\sigma^2 = 0$ and $x \bar{\Pi}(x) \rightarrow 0$ as $x \downarrow 0$, by Doney and Maller ([13], Theorem 2.1).

Use the Lévy–Itô decomposition (2.28) (with $\sigma^2 = 0$) to write X_t as

$$\begin{aligned} X_t &= \gamma t + \text{a.s.} \lim_{\varepsilon \downarrow 0} \left(\sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{\varepsilon < |\Delta X_s| \leq 1\}} - t \int_{\varepsilon < |x| \leq 1} x \Pi(dx) \right) + X_t^{(B,1)} \\ &= \gamma t + \sum_{0 < s \leq t} \Delta X_s^+ - t \int_{0 < x < 1} x \Pi(dx) - X_t^{(-)} + o_P(t). \end{aligned} \quad (4.14)$$

Here,

$$X_t^{(B,1)} = \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > 1\}} = o_P(t), \quad \text{as } t \downarrow 0,$$

because $P(|X_t^{(B,1)}| > \delta t) \leq 1 - e^{-t \bar{\Pi}(1)} \rightarrow 0$, as $t \downarrow 0$, for $\delta > 0$, and

$$X_t^{(-)} := \text{a.s.} \lim_{\varepsilon \downarrow 0} \left(\sum_{0 < s \leq t} \Delta X_s^- \mathbf{1}_{\{\varepsilon < \Delta X_s^- \leq 1\}} - t \int_{-1 \leq x < -\varepsilon} x \Pi(dx) \right).$$

In view of (4.13) and (4.14), we see that $X_t^{(-)}/t$ has a finite limit in probability as $t \downarrow 0$, and so by Doney and Maller [13], Theorem 2.1 (see also Doney [11]), the integral $\int_{(x,1)} y \Pi^{(-)}(dy)$ has a finite limit as $x \downarrow 0$. This means that $X_t^{(-)}$, and hence X_t are of bounded variation, with drift $d_X = 0$ by (4.13) and Lemma 4.1 of Doney and Maller [13].

So we can write

$$\begin{aligned} 1 + o_P(1) &= \frac{X_t}{\sup_{0 < s \leq t} \Delta X_s^+} = \frac{\sum_{0 < s \leq t} \Delta X_s^+ - \sum_{0 < s \leq t} \Delta X_s^-}{\sup_{0 < s \leq t} \Delta X_s^+} \\ &= 1 - \frac{\sum_{0 < s \leq t} \Delta X_s^-}{\sup_{0 < s \leq t} \Delta X_s^+} + o_P(1). \end{aligned} \quad (4.15)$$

From this, we see that

$$\frac{\sum_{0 < s \leq t} \Delta X_s^-}{\sup_{0 < s \leq t} \Delta X_s^+} \xrightarrow{P} 0, \quad (4.16)$$

thus by (4.1)

$$\lim_{x \downarrow 0} \frac{\int_0^x \bar{\Pi}^-(y) dy}{x \bar{\Pi}^+(x)} = 0.$$

Since $\int_0^x \bar{\Pi}^-(y) dy \geq x \bar{\Pi}^-(x)$, we have $\lim_{x \downarrow 0} \bar{\Pi}^-(x)/\bar{\Pi}^+(x) = 0$, so we have proved the forward part of Theorem 3.

For the converse, assume $\bar{\Pi}^+(x) \in SV$ at 0, X is of bounded variation with drift $d_X = 0$, and $\lim_{x \downarrow 0} \bar{\Pi}^-(x)/\bar{\Pi}^+(x) = 0$. Now $\bar{\Pi}^+(x) \in SV$ at 0 implies (4.12) by Theorem 2, and also $\int_0^x \bar{\Pi}^+(y) dy \sim x \bar{\Pi}^+(x)$ as $x \downarrow 0$. In addition, $\bar{\Pi}^-(x) = o(\bar{\Pi}^+(x))$ implies

$$\int_0^x \bar{\Pi}^-(y) dy = o\left(\int_0^x \bar{\Pi}^+(y) dy\right) = o(x \bar{\Pi}^+(x)), \quad \text{as } x \downarrow 0,$$

and then (4.16) follows as in (4.1). Thus, we get (3.13) from (4.15). \square

5. X dominating its large jump processes

In this section, we characterise divergences like²

$$\frac{X_t}{\sup_{0 < s \leq t} |\Delta X_s|} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0; \quad (5.1)$$

and similarly with $|\Delta X_s|$ replaced by ΔX_s . We think of these kinds of conditions as expressing the “dominance” of X over its largest jump processes, at small times.

These conditions will be shown to be related to the *relative stability* of the process X , and to its *attraction to normality*, as $t \downarrow 0$. Relative stability is the convergence of the normed process to a finite nonzero constant which, by rescaling of the norming function, can be taken as ± 1 . Thus, we are concerned with the property

$$\frac{X_t}{b_t} \xrightarrow{P} \pm 1, \quad \text{as } t \downarrow 0, \quad (5.2)$$

where $b_t > 0$ is a nonstochastic function. The concept is important in a variety of contexts, in particular, with reference to the stability at 0 of certain passage times for the process, as we discuss in more detail later. When X_t is replaced by $|X_t|$ in (5.1), we also bring into play the idea of X being in the *domain of attraction of the normal distribution*, as $t \downarrow 0$; that is, when there are nonstochastic functions $a_t \in \mathbb{R}$, $b_t > 0$, such that $(X_t - a_t)/b_t \xrightarrow{D} N(0, 1)$, a standard normal random variable, as $t \downarrow 0$.

Before proceeding, we quote some preliminary results, including in the next subsection a theorem originally due to Doney [11] giving necessary and sufficient conditions for X_t to

²Recall that $\bar{\Pi}(0+) = \infty$ implies $\sup_{0 < s \leq t} |\Delta X_s| > 0$ a.s. for all $t > 0$ when writing ratios like that in (5.1), and similarly for one-sided versions.

stay positive with probability approaching 1 as $t \downarrow 0$. The main result concerning relative stability is in Section 5.2, while Section 5.3 deals with 2-sided versions. The domain of attraction of the normal is needed here. Subsequential versions of the results are in Sections 5.4 and 5.5.

5.1. X staying positive near 0, in probability

Versions of truncated first and second moment functions, we will use are

$$\nu(x) = \gamma - \int_{x < |y| \leq 1} y \Pi(dy) \quad \text{and} \quad V(x) = \sigma^2 + \int_{0 < |y| \leq x} y^2 \Pi(dy), \quad x > 0. \quad (5.3)$$

Variants of $\nu(x)$ and $V(x)$ are Winsorised first and second moment functions defined by

$$A(x) = \gamma + \bar{\Pi}^+(1) - \bar{\Pi}^-(1) - \int_x^1 (\bar{\Pi}^+(y) - \bar{\Pi}^-(y)) dy \quad (5.4)$$

and

$$U(x) = \sigma^2 + 2 \int_0^x y \bar{\Pi}(y) dy \quad \text{for } x > 0. \quad (5.5)$$

$A(x)$ and $U(x)$ are continuous for $x > 0$. Using Fubini's theorem, we can show that

$$A(x) = \nu(x) + x(\bar{\Pi}^+(x) - \bar{\Pi}^-(x)) \quad (5.6)$$

and

$$U(x) = V(x) + x^2(\bar{\Pi}^+(x) + \bar{\Pi}^-(x)) = V(x) + x^2\bar{\Pi}(x). \quad (5.7)$$

These functions are finite for all $x > 0$ by virtue of property $\int_{0 < |y| \leq 1} y^2 \Pi(dy) < \infty$ of the Lévy measure Π , which further implies that $\lim_{x \downarrow 0} x^2 \bar{\Pi}(x) = 0$, and, as is easily verified,

$$\lim_{x \downarrow 0} x \nu(x) = \lim_{x \downarrow 0} x A(x) = 0. \quad (5.8)$$

Also, $\lim_{x \rightarrow \infty} A(x)/x = \lim_{x \rightarrow \infty} U(x)/x^2 = 0$. We have the obvious inequality

$$U(x) \geq \sigma^2 + x^2 \bar{\Pi}(x) \geq x^2 \bar{\Pi}(x), \quad x \geq 0.$$

This can be amplified to

$$U(x) \geq \sigma^2 + x^2 \bar{\Pi}(x-) \geq x^2 \bar{\Pi}(x-), \quad x > 0. \quad (5.9)$$

Another calculation gives (recall $\Delta \Pi(x) = \Pi\{x\}$)

$$\nu(x) - x(\Delta \Pi(x) - \Delta \Pi(-x)) = A(x) - x(\bar{\Pi}^+(x-) - \bar{\Pi}^-(x-)). \quad (5.10)$$

Lemma 5. Suppose $\sigma^2 > 0$. Then $X_t/\sqrt{t} \xrightarrow{D} N(0, \sigma^2)$ and $P(X_t > 0) \rightarrow 1/2$, as $t \downarrow 0$.

Proof. The asymptotic normality of X_t/\sqrt{t} when $\sigma^2 > 0$ is proved in Doney and Maller ([13], Theorem 2.5 and its corollary, page 760), and then $\lim_{t \downarrow 0} P(X_t > 0) = 1/2$ is immediate. \square

Next, we quote the (slightly modified) theorem originally due to Doney [11]. It shows that X remains positive with probability approaching 1 iff X dominates its large negative jumps, and explicit equivalences for this are given in terms of the functions $A(x)$, $U(x)$ and the negative tail of Π . The latter conditions reflect the positivity of X in that the function $A(x)$ remains positive for small values of x ; and $A(x)$ dominates $U(x)$ and the negative tail of Π in certain ways. Recall the notation $\Delta X_t^+ = \max(\Delta X_t, 0)$, $\Delta X_t^- = \max(-\Delta X_t, 0)$, and $(\Delta X^+)_t^{(1)} = \sup_{0 < s \leq t} \Delta X_s^+$, $(\Delta X^-)_t^{(1)} = \sup_{0 < s \leq t} \Delta X_s^-$.

Theorem 5. Suppose $\overline{\Pi}^+(0+) = \infty$.

(i) Suppose also that $\overline{\Pi}^-(0+) > 0$. Then the following are equivalent:

$$\lim_{t \downarrow 0} P(X_t > 0) = 1; \quad (5.11)$$

$$\frac{X_t}{(\Delta X^-)_t^{(1)}} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0; \quad (5.12)$$

$$\sigma^2 = 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{A(x)}{x \overline{\Pi}^-(x)} = \infty; \quad (5.13)$$

$$\lim_{x \downarrow 0} \frac{A(x)}{\sqrt{U(x) \overline{\Pi}^-(x)}} = \infty; \quad (5.14)$$

there is a nonstochastic nondecreasing function $\ell(x) > 0$, which is slowly varying at 0, such that

$$\frac{X_t}{t\ell(t)} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0. \quad (5.15)$$

(ii) Suppose X is spectrally positive, so $\overline{\Pi}^-(x) = 0$ for $x > 0$. Then (5.11) is equivalent to

$$\sigma^2 = 0 \quad \text{and} \quad A(x) \geq 0 \quad \text{for all small } x, \quad (5.16)$$

and this happens if and only if X is a subordinator. Furthermore, we then have $A(x) \geq 0$, not only for small x , but for all $x > 0$.

Remark 2. We adopt the convention that (5.12) is taken to hold when (5.11) holds but $\sup_{0 < s \leq t} \Delta X_s^- = 0$ a.s. for all small $t > 0$. This is the case when $\overline{\Pi}^-(0+) < \infty$.

Lemma 6. *If $\bar{\Pi}^-(0+) > 0$, then*

$$\limsup_{x \downarrow 0} \frac{A(x)}{\sqrt{\bar{\Pi}^-(x)}} < \infty. \quad (5.17)$$

If $\bar{\Pi}^-(0+) = 0$ and $\bar{\Pi}^+(0+) > 0$, then

$$\limsup_{x \downarrow 0} \frac{A(x)}{\sqrt{\bar{\Pi}^+(x)}} < \infty. \quad (5.18)$$

Proof of Lemma 6. (i) Assume $\bar{\Pi}^-(0+) > 0$ and, by way of contradiction, that there is a nonstochastic sequence $x_k \downarrow 0$ as $k \rightarrow \infty$ such that

$$\frac{A(x_k)}{\sqrt{\bar{\Pi}^-(x_k)}} = \frac{\gamma + \bar{\Pi}^+(1) - \bar{\Pi}^-(1) - \int_{x_k}^1 \bar{\Pi}^+(y) dy + \int_{x_k}^1 \bar{\Pi}^-(y) dy}{\sqrt{\bar{\Pi}^-(x_k)}} \rightarrow \infty.$$

Since $\bar{\Pi}^-(0+) > 0$, we deduce from this that

$$\frac{-\bar{\Pi}^-(1) + \int_{x_k}^1 \bar{\Pi}^-(y) dy}{\sqrt{\bar{\Pi}^-(x_k)}} \rightarrow \infty.$$

Thus, integrating by parts,

$$\frac{-x_k \bar{\Pi}^-(x_k) + \int_{x_k < y \leq 1} y \Pi^{(-)}(dy)}{\sqrt{\bar{\Pi}^-(x_k)}} \rightarrow \infty.$$

But by the Cauchy–Schwarz inequality,

$$\begin{aligned} \frac{(\int_{x_k < y \leq 1} y \Pi^{(-)}(dy))^2}{\bar{\Pi}^-(x_k)} &\leq \frac{\int_{x_k < y \leq 1} y^2 \Pi^{(-)}(dy) \int_{x_k < y \leq 1} \Pi^{(-)}(dy)}{\bar{\Pi}^-(x_k)} \\ &\leq \frac{\int_{x_k < |y| \leq 1} y^2 \Pi(dy) (\bar{\Pi}^-(x_k) - \bar{\Pi}^-(1))}{\bar{\Pi}^-(x_k)} \leq \int_{0 < |y| \leq 1} y^2 \Pi(dy) < \infty, \end{aligned}$$

giving a contradiction. Thus, (5.17) holds.

(ii) Alternatively, suppose $\bar{\Pi}^-(0+) = 0$ and $\bar{\Pi}^+(0+) > 0$. Then, for $0 < x < 1$,

$$\frac{A(x)}{\sqrt{\bar{\Pi}^+(x)}} = \frac{\gamma + \bar{\Pi}^+(1) - \int_x^1 \bar{\Pi}^+(y) dy}{\sqrt{\bar{\Pi}^+(x)}} \leq \frac{\gamma + \bar{\Pi}^+(1)}{\sqrt{\bar{\Pi}^+(x)}},$$

and since $\overline{\Pi}^+(0+) > 0$ the RHS is finite as $x \downarrow 0$, so (5.18) is proved. \square

Proof. Theorem 5 only differs from Theorem 1 in Doney [11] (and his remark following the theorem, regarding part (ii) of our Theorem 5) in that he assumes a priori that $\sigma^2 = 0$. Clearly, (5.11), (5.12) and (5.15) imply this by Lemma 5. (5.14) also implies $\sigma^2 = 0$. To see this, suppose on the contrary that $\sigma^2 > 0$. Then $U(x) \geq \sigma^2$ for all $x \geq 0$ and by Lemma 6, (5.17) contradicts (5.14). \square

We have the following subsequential version of Theorem 5. We omit the proof which is along the lines of Doney's proof, together with similar ideas as in Theorem 9.

Theorem 6. Suppose $\overline{\Pi}^+(0+) = \infty$.

(i) Suppose also that $\overline{\Pi}^-(0+) > 0$. Then the following are equivalent: there is a non-stochastic sequence $t_k \downarrow 0$ such that

$$P(X_{t_k} > 0) \rightarrow 1; \quad (5.19)$$

there is a nonstochastic sequence $t_k \downarrow 0$ such that

$$\frac{X_{t_k}}{(\Delta X^-)^{(1)}_{t_k}} \xrightarrow{P} \infty, \quad \text{as } k \rightarrow \infty; \quad (5.20)$$

$$\limsup_{x \downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}^-(x)}} = \infty. \quad (5.21)$$

(ii) Suppose X is spectrally positive, that is, $\overline{\Pi}^-(x) = 0$ for all $x > 0$. Then (5.19) is equivalent to $\lim_{t \downarrow 0} P(X_t > 0) \rightarrow 1$, thus to (5.16), equivalently, X_t is a subordinator, and $A(x) \geq 0$ for all $x > 0$.

Remark 3. We get equivalences for

$$\frac{X_t}{(\Delta X^+)^{(1)}_t} \xrightarrow{P} -\infty$$

(or the subsequential version) by applying Theorem 5 (or Theorem 6) with X replaced by $-X$.

In the next two subsections, we characterise when X dominates its large positive jumps and its jumps large in modulus, while remaining positive in probability, and when $|X|$ dominates its jumps large in modulus. These kinds of behaviour require more stringent conditions on X , namely, relative stability or attraction to normality, in the respective cases.

5.2. Relative stability and dominance

Recall that X is said to be *relatively stable* (RS) at 0 if (5.2) holds. X is *positively relatively stable* (PRS) at 0 if (5.2) holds with a “+” sign, and *negatively relatively stable* (NRS) at 0 if (5.2) holds with a “−” sign. In either case, the function $b_t > 0$ is regularly varying at 0 with index 1. In Griffin and Maller ([20], Proposition 2.1) it is shown (when $\bar{\Pi}(0+) = \infty$) that there is a measurable nonstochastic function $b_t > 0$ such that

$$\frac{|X_t|}{b_t} \xrightarrow{P} 1, \quad \text{as } t \downarrow 0, \quad (5.22)$$

iff $X \in RS$ at 0, equivalently, iff

$$\sigma^2 = 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{|A(x)|}{x\bar{\Pi}(x)} = \infty. \quad (5.23)$$

The following conditions characterise the convergence in (5.2) (Kallenberg [23], Theorem 15.14): for all $x > 0$,

$$\lim_{t \downarrow 0} t\bar{\Pi}(xb_t) = 0, \quad \lim_{t \downarrow 0} \frac{tA(xb_t)}{b_t} = \pm 1, \quad \lim_{t \downarrow 0} \frac{tU(xb_t)}{b_t^2} = 0. \quad (5.24)$$

Obvious modifications of these characterise convergence through a subsequence t_k in (5.2).

Next is our main result relating “one-sided” dominance to positive relative stability. The identity (2.12) supplies a key step in the proof.

Theorem 7. *Assume $\bar{\Pi}^+(0+) = \infty$. Then the following are equivalent:*

$$\frac{X_t}{(\Delta X^+)_t^{(1)}} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0; \quad (5.25)$$

$$\frac{X_t}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0; \quad (5.26)$$

$$\sigma^2 = 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{A(x)}{x\bar{\Pi}(x)} = \infty; \quad (5.27)$$

$$X \in \text{PRS} \quad \text{at } 0; \quad (5.28)$$

$$\lim_{x \downarrow 0} \frac{A(x)}{\sqrt{U(x)\bar{\Pi}(x)}} = \infty; \quad (5.29)$$

$$\lim_{x \downarrow 0} \frac{x A(x)}{U(x)} = \infty. \quad (5.30)$$

Before proving the theorem, we record the following moment formulae. Recall that \tilde{X}_t^v is defined in (2.4).

Lemma 7. *When $\bar{\Pi}^{\leftarrow}(v) < 1$ and $t > 0$:*

$$\begin{aligned} t^{-1}E\tilde{X}_t^v &= \nu(\bar{\Pi}^{\leftarrow}(v)) - \bar{\Pi}^{\leftarrow}(v)(\Delta\Pi(\bar{\Pi}^{\leftarrow}(v)) - \Delta\Pi(-\bar{\Pi}^{\leftarrow}(v))) \\ &= A(\bar{\Pi}^{\leftarrow}(v)) - \bar{\Pi}^{\leftarrow}(v)(\bar{\Pi}^+(\bar{\Pi}^{\leftarrow}(v)-) - \bar{\Pi}^-(\bar{\Pi}^{\leftarrow}(v)-)). \end{aligned} \quad (5.31)$$

For all $t > 0$, $v > 0$,

$$E(\tilde{X}_t^v)^2 = t\left(\sigma^2 + \int_{|x| < \bar{\Pi}^{\leftarrow}(v)} x^2 \Pi(dx)\right) + (E\tilde{X}_t^v)^2. \quad (5.32)$$

Proof. Let $(U_t)_{t \geq 0}$ be a Lévy process with triplet $(\gamma_U, \sigma_U^2, \Pi_U)$. Provided the participating integrals are finite (see Example 25.11 in Sato [47]), for instance, we have

$$EU_t = t\left(\gamma_U + \int_{|y| > 1} y \Pi_U(dy)\right) \quad \text{and} \quad E(U_t)^2 = t\left(\sigma_U^2 + \int_{\mathbb{R}_*} y^2 \Pi_U(dy)\right) + (EU_t)^2.$$

Apply these to \tilde{X}_t^v with triplet as in (2.5) to get, when $\bar{\Pi}^{\leftarrow}(v) < 1$ and $t > 0$,

$$\begin{aligned} t^{-1}E\tilde{X}_t^v &= \gamma - \int_{\bar{\Pi}^{\leftarrow}(v) \leq |x| \leq 1} x \Pi(dx) \\ &= \gamma - \int_{\bar{\Pi}^{\leftarrow}(v) < |x| \leq 1} x \Pi(dx) - \bar{\Pi}^{\leftarrow}(v)(\Delta\Pi(\bar{\Pi}^{\leftarrow}(v)) - \Delta\Pi(-\bar{\Pi}^{\leftarrow}(v))), \end{aligned}$$

which gives the first equation in (5.31). For the second equation in (5.31), use (5.10). (5.32) is proved similarly. \square

Proof of Theorem 7. Assume $\bar{\Pi}^+(0+) = \infty$ throughout.

Case (i). Suppose $\bar{\Pi}^-(0+) > 0$.

(5.25) \implies (5.26): Assume (5.25). This implies $\lim_{t \downarrow 0} P(X_t > 0) = 1$, so by Theorem 5, (5.12) holds. (5.12) together with (5.25) implies (5.26), because $|\widehat{\Delta X}_t^{(1)}| = \max((\Delta X^+)_t^{(1)}, (\Delta X^-)_t^{(1)})$.

(5.26) \implies (5.27): Assume (5.26). Then $\lim_{t \downarrow 0} P(X_t > 0) = 1$, so (5.13) holds. Since $\bar{\Pi}^-(0+) > 0$, (5.13) implies $\lim_{x \downarrow 0} A(x)/x = \infty$; in particular, $A(x) > 0$ for all small x . Since $\lim_{t \downarrow 0} P(X_t > 0) = 1$, Lemma 5 in Doney [11] gives

$$U(x) \leq 3xA(x) \quad \text{for all small } x, x \leq x_0, \text{ say.} \quad (5.33)$$

Without loss of generality, assume $x_0 < 1$.

Note that (5.26) also implies

$$\frac{{}^{(1)}\widetilde{X}_t}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{P} \infty, \quad \text{as } x \downarrow 0$$

(recall (2.3)), so we have

$$\lim_{t \downarrow 0} P({}^{(1)}\widetilde{X}_t \leq a |\widetilde{\Delta X}_t^{(1)}|) = 0 \quad \text{for some } a > 0. \quad (5.34)$$

(In fact, this holds for all $a > 0$. But it will be enough to assume (5.34).) Without loss of generality, take $a \leq 1$.

We will abbreviate $\overline{\Pi}^\leftarrow(v)$ to y_v throughout this proof. Then by (2.12), we can write

$$P({}^{(1)}\widetilde{X}_t \leq a |\widetilde{\Delta X}_t^{(1)}|) = \int_0^\infty P(\widetilde{X}_t^v + \widetilde{G}_t^v \leq a y_v) P(\mathfrak{E} \in t dv), \quad (5.35)$$

where $\mathfrak{E} = \mathfrak{S}_1$ is a unit exponential r.v. By (2.7) and (2.8),

$$\begin{aligned} |E\widetilde{G}_t^v| &= y_v |EY_{t\kappa^+(v)}^+ - EY_{t\kappa^-(v)}^-| \\ &\leq t y_v (\overline{\Pi}(y_v -) - v) \frac{\Delta\Pi(y_v) + \Delta\Pi(-y_v)}{\Delta\overline{\Pi}(y_v)} \mathbf{1}_{\{\Delta\overline{\Pi}(y_v) \neq 0\}} \\ &\leq t y_v \overline{\Pi}(y_v -) \leq t U(y_v)/y_v \quad (\text{by (5.9)}) \\ &\leq 3tA(y_v) \quad (\text{by (5.33)}) \end{aligned} \quad (5.36)$$

and similarly

$$\text{Var}(\widetilde{G}_t^v) \leq t y_v^2 \overline{\Pi}(y_v -). \quad (5.37)$$

With x_0 as in (5.33), keep $v \geq \overline{\Pi}(x_0)$, so $y_v \leq x_0 < 1$. Then

$$\begin{aligned} E\widetilde{X}_t^v &= t(A(y_v) - y_v(\overline{\Pi}^+(y_v -) - \overline{\Pi}^-(y_v -))) \quad (\text{by (5.31)}) \\ &\leq t(A(y_v) + y_v \overline{\Pi}^-(y_v -)) \\ &\leq 4tA(y_v) \quad (\text{by (5.9) and (5.33)}). \end{aligned} \quad (5.38)$$

Apply (5.36) and (5.38) to obtain from (5.35)

$$\begin{aligned} P({}^{(1)}\widetilde{X}_t \leq a |\widetilde{\Delta X}_t^{(1)}|) \\ \geq \int_{\overline{\Pi}(x_0)}^\infty P(\widetilde{X}_t^v - E\widetilde{X}_t^v + \widetilde{G}_t^v - E\widetilde{G}_t^v \leq a y_v - 7tA(y_v)) P(\mathfrak{E} \in t dv). \end{aligned} \quad (5.39)$$

For $t > 0$ and a as in (5.34) define

$$b_t := \sup \left\{ x > 0 : \frac{A(x)}{x} > \frac{a^2}{56t} \right\}, \quad (5.40)$$

with $b_0 := 0$. Recall that $\lim_{x \downarrow 0} A(x)/x = \infty$, $\lim_{x \rightarrow \infty} A(x)/x = 0$, and $A(x)$ is continuous. So $0 < b_t < \infty$, b_t is strictly increasing, $b_t \downarrow 0$ as $t \downarrow 0$, and

$$\frac{tA(b_t)}{b_t} = \frac{a^2}{56}. \quad (5.41)$$

Assume t is small enough for $b_t \leq x_0$ and keep $v < \bar{\Pi}(b_t)$. Then $y_v \geq b_t$, and so $7tA(y_v) \leq a^2 y_v / 8$ by definition of b_t . This implies $7tA(y_v) \leq a y_v / 2$. Thus, by Chebyshev's inequality and (5.39)

$$\begin{aligned} P^{(1)}(\tilde{X}_t \leq a|\widetilde{\Delta X}_t^{(1)}|) &\geq \int_{\bar{\Pi}(x_0)}^{\bar{\Pi}(b_t)} P(\tilde{X}_t^v - E\tilde{X}_t^v + \tilde{G}_t^v - E\tilde{G}_t^v \leq a y_v / 2) P(\mathfrak{E} \in t dv) \\ &\geq \int_{\bar{\Pi}(x_0)}^{\bar{\Pi}(b_t)} \left(1 - \frac{4(\text{Var}(\tilde{X}_t^v) + \text{Var}(\tilde{G}_t^v))}{a^2 y_v^2} \right) P(\mathfrak{E} \in t dv). \end{aligned} \quad (5.42)$$

Also

$$\begin{aligned} \text{Var}(\tilde{X}_t^v) + \text{Var}(\tilde{G}_t^v) &\leq t(V(y_v) + y_v^2 \bar{\Pi}(y_v -)) \quad (\text{by (5.32) and (5.37)}) \\ &\leq 2tU(y_v) \quad (\text{see (5.7) and (5.9)}) \\ &\leq 6t y_v A(y_v) \quad (\text{by (5.33), since } y_v \leq x_0) \\ &\leq a^2 y_v^2 / 8. \end{aligned} \quad (5.43)$$

The last inequality holds since $y_v \geq b_t$. Hence, from (5.42),

$$\begin{aligned} P^{(1)}(\tilde{X}_t \leq a|\widetilde{\Delta X}_t^{(1)}|) &\geq t \int_{\bar{\Pi}(x_0)}^{\bar{\Pi}(b_t)} e^{-tv} dv / 2 \\ &= e^{-t\bar{\Pi}(x_0)} (1 - e^{-t(\bar{\Pi}(b_t) - \bar{\Pi}(x_0))}) / 2. \end{aligned}$$

Since the left-hand side tends to 0 as $t \downarrow 0$ by (5.34), we see from (5.41) that

$$t\bar{\Pi}(b_t) = \frac{a^2 b_t \bar{\Pi}(b_t)}{56A(b_t)} \rightarrow 0, \quad \text{as } t \downarrow 0. \quad (5.44)$$

Now take $\lambda > 1$ and write, by (5.41),

$$\frac{b_{\lambda t}}{b_t} = \frac{56\lambda t A(b_{\lambda t})}{a^2 b_t} = \frac{56\lambda t A(b_t)}{a^2 b_t} + \frac{56\lambda t (A(b_{\lambda t}) - A(b_t))}{a^2 b_t}$$

$$\begin{aligned}
 &= \lambda + \frac{56\lambda t \int_{b_t}^{b_{\lambda t}} (\bar{\Pi}^+(y) - \bar{\Pi}^-(y)) dy}{a^2 b_t} \\
 &= \lambda + O(t\bar{\Pi}(b_t)) \left(\frac{b_{\lambda t} - b_t}{b_t} \right) \\
 &= \lambda + o\left(\frac{b_{\lambda t}}{b_t}\right).
 \end{aligned}$$

Thus, b_t is regularly varying with index 1 as $t \downarrow 0$. Also, (5.44) implies $A(b_t)/b_t\bar{\Pi}(b_t) \rightarrow \infty$ as $t \downarrow 0$. From those we obtain (5.27) as follows. Given $x > 0$ choose $t = t(x)$ so that $b_{t-} \leq x \leq b_{t+}$. Then, for any $\varepsilon \in (0, 1)$, $b_{t(1-\varepsilon)} \leq x \leq b_{t(1+\varepsilon)}$, while $b_{t(1+\varepsilon)} \sim (1+\varepsilon)b_t \sim (1+\varepsilon)b_{(1-\varepsilon)t/(1-\varepsilon)}$ as $t \downarrow 0$. So

$$\begin{aligned}
 A(x) &= A(b_{t(1-\varepsilon)}) + \int_{b_{t(1-\varepsilon)}}^x (\bar{\Pi}^+(y) - \bar{\Pi}^-(y)) dy \\
 &\geq A(b_{t(1-\varepsilon)}) - b_{t(1+\varepsilon)}\bar{\Pi}(b_{t(1-\varepsilon)}) \\
 &\geq (1+o(1))A(b_{t(1-\varepsilon)}) \quad (\text{by (5.44)}).
 \end{aligned}$$

Hence, as $x \downarrow 0$,

$$\frac{A(x)}{x\bar{\Pi}(x)} \geq \frac{(1+o(1))A(b_{t(1-\varepsilon)})}{b_{t(1-\varepsilon)}\bar{\Pi}(b_{t(1-\varepsilon)})} \times \frac{b_{t(1-\varepsilon)}}{b_{t(1+\varepsilon)}} \rightarrow \infty, \quad (5.45)$$

and (5.27) is proved.

(5.27) \iff (5.28) is in Theorem 2.2 of Doney and Maller [13].

(5.28) \implies (5.29): (5.28) implies $A(b_t)/\sqrt{U(b_t)\bar{\Pi}(b_t)} \rightarrow \infty$ by (5.24) and then (5.29) follows from the regular variation of b_t (noted prior to (5.22)), by similar arguments as we used in proving (5.45) from (5.44).

(5.29) \iff (5.30): (5.29) implies $X \in \text{PRS}$ at 0, so $b_t A(b_t)/U(b_t) \rightarrow \infty$ by (5.24), and b_t is regularly varying with index 1 at 0. Then (5.30) follows by similar arguments as we used in proving (5.45) from (5.44). Conversely, (5.30) implies (5.29) because $U(x) \geq x^2\bar{\Pi}(x)$.

In the reverse direction, we will show that (5.29) \implies (5.27) \implies (5.26) \implies (5.25).

(5.29) \implies (5.27): (5.29) implies (5.14), hence $\sigma^2 = 0$ by Theorem 5. Then (5.27) follows from (5.29) since $U(x) \geq x^2\bar{\Pi}(x)$.

(5.27) \implies (5.26): Assume (5.27). This implies $X \in \text{PRS}$, so $X_t/b_t \xrightarrow{P} +1$ as $t \downarrow 0$ for some $b_t > 0$. By (5.24), $\lim_{t \downarrow 0} t\bar{\Pi}(\varepsilon b_t) = 0$ for all $\varepsilon > 0$. This implies

$$P\left(\sup_{0 < s \leq t} |\Delta X_s| > \varepsilon b_t\right) = 1 - e^{-t\bar{\Pi}(\varepsilon b_t)} \rightarrow 0,$$

thus $\sup_{0 < s \leq t} |\Delta X_s|/b_t \xrightarrow{P} 0$ as $t \downarrow 0$. So we get (5.26).

(5.26) \implies (5.25) is true since $|\widetilde{\Delta X}_t^{(1)}| \geq (\Delta X^+)_t^{(1)}$. So we have shown the equivalence of (5.25)–(5.30) for case (i).

Case (ii). Suppose $\overline{\Pi}^-(0+) = 0$. By part (ii) of Theorem 5, each of (5.25)–(5.28) implies X is a subordinator (with drift) and $A(x) \geq 0$ for all $x \geq 0$. (5.25) and (5.26) are the same thing in this case.

(5.26) \implies (5.27): Assume (5.26). Since X is a subordinator, we can write

$$A(x) = d_X + \int_0^x \overline{\Pi}^+(y) dy, \quad x \geq 0,$$

where $d_X \geq 0$ is the drift of X and $\int_0^x \overline{\Pi}^+(y) dy < \infty$. The latter implies $\lim_{x \downarrow 0} x \overline{\Pi}^+(x) = 0$. Of course $\sigma^2 = 0$ and if $d_X > 0$ then (5.27) clearly holds. So suppose $d_X = 0$. As in (5.38), we get $E\tilde{X}_t^v \leq tA(y_v)$ and (5.36) and (5.37) remain true. Since $\overline{\Pi}^+(0+) = \infty$,

$$\lim_{x \downarrow 0} \frac{A(x)}{x} \geq \int_0^1 \liminf_{x \downarrow 0} \overline{\Pi}^+(xy) dy = \infty.$$

Define b_t again by (5.40). Then the same working as in case (i) gives $t\overline{\Pi}(b_t) \rightarrow 0$ and b_t regularly varying with index 1, so again we get (5.27).

(5.27) \iff (5.28) is in Theorem 2.2 of Doney and Maller [13] in this case also; their theorem only requires $\overline{\Pi}(0+) > 0$.

The remaining equivalences in case (ii) follow exactly as in case (i). This completes the proof of Theorem 7. \square

The *domain of attraction of the normal distribution*, as $t \downarrow 0$, appears in the next result, which is a corollary to Theorem 7. We say $X \in D(N)$ at 0 if there are functions $a_t \in \mathbb{R}$, $b_t > 0$, such that $(X_t - a_t)/b_t \xrightarrow{D} N(0, 1)$ (a standard normal random variable) as $t \downarrow 0$. If a_t may be taken as 0, we write $X \in D_0(N)$ (no centering required). The following condition characterises the domain of attraction of the normal at 0 (Doney and Maller [13], Theorem 2.5):

$$\lim_{x \downarrow 0} \frac{U(x)}{x^2 \overline{\Pi}(x)} = \infty; \quad (5.46)$$

in fact, $D(N)$ (at 0) equals $D_0(N)$ (at 0) (Maller and Mason [38], Theorem 2.4). A characterisation for $D_0(N)$ at 0 (equivalent to (5.46)) is

$$\lim_{x \downarrow 0} \frac{U(x)}{x|A(x)| + x^2 \overline{\Pi}(x)} = \infty. \quad (5.47)$$

The following conditions are also equivalent to $X_t/b_t \xrightarrow{D} N(0, 1)$ (Kallenberg [23], Theorem 15.14): for all $x > 0$,

$$\lim_{t \downarrow 0} t\overline{\Pi}(xb_t) = 0, \quad \lim_{t \downarrow 0} \frac{tA(xb_t)}{b_t} = 0, \quad \lim_{t \downarrow 0} \frac{tU(xb_t)}{b_t^2} = 1. \quad (5.48)$$

Obvious modifications of these characterise the convergence $X_t/b_t \xrightarrow{D} N(0, 1)$ through a subsequence $t_k \downarrow 0$.

Corollary 3 (Corollary to Theorem 7). *Assume $\bar{\Pi}^+(0+) = \infty$. Then the following are equivalent:*

$$\text{there is a nonstochastic function } c_t > 0 \text{ such that } \frac{V_t}{c_t} \xrightarrow{P} 1, \quad \text{as } t \downarrow 0; \quad (5.49)$$

$$\frac{V_t}{\sup_{0 < s \leq t} |\Delta X_s|^2} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0; \quad (5.50)$$

X is in the domain of attraction of the normal distribution, as $t \downarrow 0$.

Proof. V_t is a subordinator with drift $d_V = \sigma^2$ and Lévy measure Π_V , where $\bar{\Pi}_V(x) = \bar{\Pi}(\sqrt{x})\mathbf{1}_{\{x>0\}}$. Let the triplet of V_t be $(\gamma_V, 0, \Pi_V(\cdot))$. Then $d_V = \gamma_V + \int_0^1 y \Pi_V(dy)$. Thus, in obvious notation

$$\begin{aligned} A_V(x) &= \gamma_V + \bar{\Pi}_V(1) - \int_x^1 \bar{\Pi}_V(y) dy = d_V + \int_0^x \bar{\Pi}_V(y) dy \\ &= \sigma^2 + 2 \int_0^{\sqrt{x}} y \bar{\Pi}(y) dy = U(\sqrt{x}), \quad x > 0. \end{aligned}$$

Hence,

$$\frac{A_V(x)}{x \bar{\Pi}_V(x)} = \frac{U(\sqrt{x})}{(\sqrt{x})^2 \bar{\Pi}(\sqrt{x})}$$

tends to ∞ iff (5.46) holds. By Theorem 7 these are equivalent to (5.49) and (5.50), and (5.46) characterises the domain of attraction of the normal, as noted. \square

Remark 4. (i) Another interesting kind of “self-normalisation” of a Lévy process is to divide X_t by $\sqrt{V_t}$, possibly after removal of one or the other kind of maximal jump. See, for example, Maller and Mason [36, 39]. Our methods can be used to extend these results in a variety of directions, but we omit further details here.

(ii) Relative stability of X is directly related to the stability of the “one-sided” and “two-sided” passage times over power law boundaries defined by

$$\bar{T}_b(r) := \inf\{t \geq 0 : X_t > rt^b\}, \quad r \geq 0,$$

and

$$T_b^*(r) := \inf\{t \geq 0 : |X_t| > rt^b\}, \quad r \geq 0,$$

when³ $0 \leq b < 1$. Griffin and Maller [20] show that, then, $\bar{T}_b(r)$ is relatively stable as $r \downarrow 0$, in the sense that $\bar{T}_b(r)/C(r) \xrightarrow{P} 1$ as $r \downarrow 0$ for a nonstochastic function $C(r) > 0$, iff $X \in$

³Griffin and Maller [20] show that relative stability of $\bar{T}_b(r)$ or $T_b^*(r)$ cannot obtain when $b \geq 1$.

PRS, while $T_b^*(r)$ is relatively stable as $r \downarrow 0$, in the sense that $T_b^*(r)/C(r) \xrightarrow{P} 1$ as $r \downarrow 0$ for a nonstochastic function $C(r) > 0$, iff $X \in RS$. Further connections made in Griffin and Maller [20] are that $X \in PRS$ iff $\widetilde{X}_t := \sup_{0 < s \leq t} X_s$ is relatively stable, while $X \in RS$ iff $X_t^* := \sup_{0 < s \leq t} |X_s|$ is relatively stable. Auxiliary results are (i) there is a nonstochastic function $b_t^* > 0$ and constants $0 < c_1 < c_2 < \infty$ such that $\lim_{t \downarrow 0} P(c_1 < |X_t|/b_t^* < c_2) = 1$ iff $X \in RS$, and (ii) there is a nonstochastic function $b_t^\dagger > 0$ such that each sequence $t_k \downarrow 0$ contains a subsequence $t_{k'} \downarrow 0$ with $|X_{t_{k'}}|/b_{t_{k'}}^\dagger \xrightarrow{P} c'$, where $0 < |c'| < \infty$, iff $X \in RS$. See also Griffin and Maller [19].

5.3. Relative stability, attraction to normality and dominance

The next theorems look at two-sided results, concerning stability and dominance of $|X|$. Now the domain of attraction of the normal enters as an alternative to relative stability.

Theorem 8. Assume $\overline{\Pi}(0+) = \infty$. Then the following are equivalent:

$$\frac{|X_t|}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0; \quad (5.51)$$

$$\lim_{x \downarrow 0} \frac{x|A(x)| + U(x)}{x^2 \overline{\Pi}(x)} = \infty; \quad (5.52)$$

$$\lim_{x \downarrow 0} \frac{U(x)}{x|A(x)| + x^2 \overline{\Pi}(x)} = +\infty, \quad \text{or} \quad \lim_{x \downarrow 0} \frac{|A(x)|}{x \overline{\Pi}(x)} = +\infty; \quad (5.53)$$

$$X \in D_0(N) \cup RS \quad \text{at } 0. \quad (5.54)$$

Proof. Assume $\overline{\Pi}(0+) = \infty$. (5.51) \implies (5.52): Assume (5.51). This implies

$$\frac{|^{(1)}\widetilde{X}_t|}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{P} \infty, \quad \text{as } t \downarrow 0,$$

so we have

$$\lim_{t \downarrow 0} P(|^{(1)}\widetilde{X}_t| \leq a |\widetilde{\Delta X}_t^{(1)}|) = 0 \quad \text{for some } a > 0. \quad (5.55)$$

Without loss of generality take $a \leq 1$.

We again abbreviate $\overline{\Pi}^\leftarrow(v)$ to y_v throughout. Then by (2.12), we can write

$$P(|^{(1)}\widetilde{X}_t| \leq a |\widetilde{\Delta X}_t^{(1)}|) = \int_0^\infty P(|\widetilde{X}_t^v + \widetilde{G}_t^v| \leq a y_v) P(\mathfrak{E} \in t dv). \quad (5.56)$$

By (5.36), we have

$$|E\widetilde{G}_t^v| \leq t y_v \overline{\Pi}(y_v -) \leq t U(y_v)/y_v, \quad (5.57)$$

and (5.37) remains true. Also, as in (5.38),

$$\begin{aligned} |E\tilde{X}_t^v| &= t|A(y_v) - y_v(\bar{\Pi}^+(y_v-) - \bar{\Pi}^-(y_v-))| \\ &\leq t(|A(y_v)| + y_v\bar{\Pi}(y_v-)) \\ &\leq t(|A(y_v)| + U(y_v)/y_v). \end{aligned} \quad (5.58)$$

Apply (5.57) and (5.58) to obtain from (5.56)

$$\begin{aligned} P(|^{(1)}\tilde{X}_t| \leq a|\widetilde{\Delta X}_t^{(1)}|) &\geq \int_0^\infty P(|\tilde{X}_t^v - E\tilde{X}_t^v + \tilde{G}_t^v - E\tilde{G}_t^v| \leq ay_v - |E\tilde{X}_t^v| - |E\tilde{G}_t^v|)P(\mathfrak{E} \in t dv) \\ &\geq \int_0^\infty P(|\tilde{X}_t^v - E\tilde{X}_t^v + \tilde{G}_t^v - E\tilde{G}_t^v| \leq ay_v - 2t(|A(y_v)| + U(y_v)/y_v))P(\mathfrak{E} \in t dv). \end{aligned} \quad (5.59)$$

For $t > 0$, define

$$b_t := \sup \left\{ x > 0 : \frac{x|A(x)| + U(x)}{x^2} > \frac{a^2}{56t} \right\}, \quad (5.60)$$

with $b_0 := 0$. Since $\bar{\Pi}(0+) = \infty$, we have $\lim_{x \downarrow 0} (x|A(x)| + U(x))/x^2 = \infty$. In addition, $\lim_{x \rightarrow \infty} (x|A(x)| + U(x))/x^2 = 0$. Then $0 < b_t < \infty$, b_t is strictly increasing, $b(t) \downarrow 0$ as $t \downarrow 0$, and

$$\frac{t(b_t|A(b_t)| + U(b_t))}{b_t^2} = \frac{a^2}{56}, \quad t > 0. \quad (5.61)$$

Now keep $v < \bar{\Pi}(b_t)$. Then $y_v \geq b_t$, and so

$$t(|A(y_v)| + U(y_v)/y_v) \leq \frac{a^2 y_v}{56} \leq \frac{ay_v}{4},$$

by definition of b_t . Thus, by Chebyshev's inequality and (5.59)

$$\begin{aligned} P(|^{(1)}\tilde{X}_t| \leq a|\widetilde{\Delta X}_t^{(1)}|) &\geq \int_0^{\bar{\Pi}(b_t)} P(|\tilde{X}_t^v - E\tilde{X}_t^v + \tilde{G}_t^v - E\tilde{G}_t^v| \leq ay_v/2)P(\mathfrak{E} \in t dv) \\ &\geq \int_0^{\bar{\Pi}(b_t)} \left(1 - \frac{4(\text{Var}(\tilde{X}_t^v) + \text{Var}(\tilde{G}_t^v))}{a^2 y_v^2} \right) P(\mathfrak{E} \in t dv). \end{aligned}$$

Also, as in (5.43),

$$\text{Var}(\tilde{X}_t^v) + \text{Var}(\tilde{G}_t^v) \leq a^2 y_v^2 / 8,$$

giving

$$P(|^{(1)}\tilde{X}_t| \leq a|\widetilde{\Delta X}_t^{(1)}|) \geq t \int_0^{\bar{\Pi}(b_t)} e^{-tv} dv / 2 = (1 - e^{-t\bar{\Pi}(b_t)})/2. \quad (5.62)$$

Since the left-hand side tends to 0 as $t \downarrow 0$ by (5.55) we see that

$$t\bar{\Pi}(b_t) = \frac{a^2 b_t^2 \bar{\Pi}(b_t)}{56(b_t|A(b_t)| + U(b_t))} \rightarrow 0, \quad \text{as } t \downarrow 0. \quad (5.63)$$

We need to replace b_t by a continuous variable $x \downarrow 0$ in this. By (5.61), for $\lambda > 1$ and $t > 0$

$$\begin{aligned} \frac{b_{t\lambda}^2}{b_t^2} &= \frac{56t\lambda(b_{t\lambda}|A(b_{t\lambda})| + U(b_{t\lambda}))}{a^2 b_t^2} \\ &= \frac{56t\lambda(b_{t\lambda}|A(b_t)| + U(b_t))}{a^2 b_t^2} + \frac{56t\lambda b_{t\lambda}(|A(b_{t\lambda})| - |A(b_t)|)}{a^2 b_t^2} \\ &\quad + \frac{56t\lambda(U(b_{t\lambda}) - U(b_t))}{a^2 b_t^2} \\ &\leq \lambda + \frac{56t\lambda(b_{t\lambda} - b_t)|A(b_t)|}{a^2 b_t^2} + \frac{56t\lambda b_{t\lambda}(b_{t\lambda} - b_t)\bar{\Pi}(b_t)}{a^2 b_t^2} + \frac{56t\lambda(b_{t\lambda}^2 - b_t^2)\bar{\Pi}(b_t)}{a^2 b_t^2}. \end{aligned} \quad (5.64)$$

Observe that $56t\lambda(b_{t\lambda} - b_t)|A(b_t)|/a^2 b_t^2 \leq \lambda(b_{t\lambda} - b_t)/b_t$. Since $t\bar{\Pi}(b_t) = o(1)$, (5.64) implies

$$\frac{b_{t\lambda}^2}{b_t^2} \leq \lambda + \lambda \left(\frac{b_{t\lambda}}{b_t} - 1 \right) + o \left(\frac{b_{t\lambda}}{b_t} \right) \leq \lambda + \lambda \frac{b_{t\lambda}}{b_t} + o \left(\frac{b_{t\lambda}}{b_t} \right).$$

From this, we deduce that $\limsup_{t \downarrow 0} b_{t\lambda}/b_t < \infty$.

Now return to (5.63) and take $x > 0$. Choose $t = t(x)$ such that $b_t \leq x \leq b_{t\lambda}$, $\lambda > 1$. It is shown in Klass and Wittmann [30] that the function $x|A(x)| + U(x)$ is nondecreasing⁴ in $x > 0$. Thus,

$$\frac{x|A(x)| + U(x)}{x^2 \bar{\Pi}(x)} \geq \frac{b_t|A(b_t)| + U(b_t)}{b_t^2 \bar{\Pi}(b_t)} \times \frac{b_t^2}{b_{t\lambda}^2}.$$

The first factor on the right tends to ∞ as $t \downarrow 0$ by (5.63), and $\liminf_{t \downarrow 0} b_t/b_{t\lambda} > 0$, so we get (5.52).

(5.52) \iff (5.53) is proved in Lemma 4 of Doney and Maller [14].

(5.53) \implies (5.54): Assume (5.53). If $\sigma^2 > 0$ then by Lemma 5, $X \in D_0(N)$ hence $X \in D_0(N) \cup RS$. So suppose $\sigma^2 = 0$. Then the left-hand side of (5.53) is equivalent to

⁴Klass and Wittmann prove this for versions of A and U defined for distribution functions. But their proof is easily modified to apply to the present A and U .

$X \in D_0(N)$ at 0 by (5.47), and the right-hand side of (5.53) is equivalent to $X_t \in RS$ at 0 by (5.23). Thus again, $X \in D_0(N) \cup RS$.

(5.54) \implies (5.51): Finally, if $X \in D_0(N) \cup RS$ then $X_t/b_t \xrightarrow{D} N(0,1)$ for some $b_t > 0$ with $\widetilde{\Delta X}_t^{(1)} = o_P(b_t)$ or $X_t/c_t \xrightarrow{P} \pm 1$ for some $c_t > 0$ with $\widetilde{\Delta X}_t^{(1)} = o_P(c_t)$, and in either case (5.51) holds. This completes Theorem 8. \square

5.4. Subsequential relative stability and dominance

We say that X is *subsequentially relatively stable* (SRS) at 0 if there are nonstochastic sequences $t_k \downarrow 0$ and $b_k > 0$ such that

$$\frac{X_{t_k}}{b_k} \xrightarrow{P} \pm 1, \quad \text{as } k \rightarrow \infty. \quad (5.65)$$

Define positive and negative subsequential relative stability (PSRS and NSRS) in the obvious ways.

Theorem 9. Assume $\overline{\Pi}^+(0+) = \infty$. Then the following are equivalent: there is a nonstochastic sequence $t_k \downarrow 0$ such that

$$\frac{X_{t_k}}{|\widetilde{\Delta X}_{t_k}^{(1)}|} \xrightarrow{P} \infty, \quad \text{as } k \rightarrow \infty; \quad (5.66)$$

there is a nonstochastic sequence $t_k \downarrow 0$ such that

$$\frac{X_{t_k}}{(\Delta X^+)_{t_k}^{(1)}} \xrightarrow{P} \infty, \quad \text{as } k \rightarrow \infty; \quad (5.67)$$

$$X \in \text{PSRS at } 0; \quad (5.68)$$

$$\limsup_{x \downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}(x)}} = \infty; \quad (5.69)$$

$$\limsup_{x \downarrow 0} \frac{x A(x)}{U(x)} = \infty. \quad (5.70)$$

Proof. Assume $\overline{\Pi}^+(0+) = \infty$. Each of (5.66)–(5.70) implies $\sigma^2 = 0$; by Lemma 5 in the case of (5.66) and (5.67), by Lemma 6 in the case of (5.69), and by (5.8) and $U(x) \geq \sigma^2$, in the case of (5.70). So we assume throughout that $\sigma^2 = 0$.

(5.66) \iff (5.67): clearly, (5.66) implies (5.67). Conversely, assume (5.67). From (5.20), we have that $X_{t_k}/(\Delta X^-)_{t_k}^{(1)} \xrightarrow{P} \infty$, as $k \rightarrow \infty$, when $\lim_{k \rightarrow \infty} P(X_{t_k} > 0) = 1$. Together with (5.67) and $|\widetilde{\Delta X}_t^{(1)}| = \max((\Delta X^+)^{(1)}_t, (\Delta X^-)^{(1)}_t)$, this implies (5.66).

(5.69) \iff (5.70): Assume (5.69), so there is a nonstochastic sequence $x_k \downarrow 0$ such that

$$\frac{A(x_k)}{\sqrt{U(x_k)\overline{\Pi}(x_k)}} \rightarrow \infty, \quad \text{as } k \rightarrow \infty.$$

Define

$$t_k = \frac{1}{A(x_k)} \sqrt{\frac{U(x_k)}{\overline{\Pi}(x_k)}}.$$

Then

$$t_k \overline{\Pi}(x_k) = \frac{\sqrt{U(x_k)\overline{\Pi}(x_k)}}{A(x_k)} \rightarrow 0$$

and so, since $\overline{\Pi}(0+) > 0$, $t_k \rightarrow 0$. Also

$$\frac{U(x_k)}{t_k A^2(x_k)} = \frac{1}{A(x_k)} \sqrt{\overline{\Pi}(x_k)U(x_k)} \rightarrow 0.$$

Let $b_k = t_k A(x_k)$, then

$$\frac{b_k}{x_k} = \frac{t_k A(x_k)}{x_k} = \sqrt{\frac{U(x_k)}{x_k^2 \overline{\Pi}(x_k)}} \geq 1.$$

Now since $b_k \geq x_k$ we have

$$\begin{aligned} \frac{t_k U(b_k)}{b_k^2} &= \frac{U(x_k)}{t_k A^2(x_k)} + \frac{2t_k \int_{x_k}^{b_k} y \overline{\Pi}(y) dy}{b_k^2} \\ &\leq o(1) + O(t_k \overline{\Pi}(x_k)) = o(1). \end{aligned}$$

This implies $t_k U(xb_k)/b_k^2 = o(1)$ for all $x \in (0, 1]$, hence

$$\lim_{k \rightarrow \infty} t_k \overline{\Pi}(xb_k) = 0 \quad \text{for all } x \in (0, 1], \quad (5.71)$$

because $U(x) \geq x^2 \overline{\Pi}(x)$. But then since $\overline{\Pi}$ is nonincreasing, (5.71) holds for all $x > 0$. Thus, also, for $x > 1$,

$$\frac{t_k U(xb_k)}{b_k^2} = \frac{t_k U(b_k)}{b_k^2} + O(t_k \overline{\Pi}(b_k)) = o(1). \quad (5.72)$$

Again since $b_k \geq x_k$, we can write

$$\frac{t_k A(b_k)}{b_k} = 1 + \frac{t_k \int_{x_k}^{b_k} (\overline{\Pi}^+(y) - \overline{\Pi}^-(y)) dy}{b_k} = 1 + O(t_k \overline{\Pi}(x_k)) = 1 + o(1). \quad (5.73)$$

(5.72) and (5.73), hence (5.69), imply (5.70). Conversely, (5.70) implies (5.69) because $U(x) \geq x^2 \bar{\Pi}(x)$.

(5.69) \iff (5.68): (5.69) implies (5.71)–(5.73), as just shown, and these together imply (5.65) (with a “+” sign) by the subsequential version of (5.24). Thus, (5.68) holds. Conversely, assuming (5.68), we get (5.71)–(5.73) by the subsequential version of (5.24). But then (5.69) holds because

$$\frac{A(b_k)}{\sqrt{U(b_k) \bar{\Pi}(b_k)}} = \frac{t_k A(b_k)}{b_k} \sqrt{\left(\frac{b_k^2}{t_k U(b_k)}\right) \left(\frac{1}{t_k \bar{\Pi}(b_k)}\right)} \rightarrow \infty.$$

So we have proved the equivalence of (5.68)–(5.70).

(5.66) \implies (5.69): Assume (5.66).

Case (i). Suppose $\bar{\Pi}^-(0+) > 0$. Then, using Theorem 6, we have $\lim_{k \rightarrow \infty} P(X_{t_k} > 0) = 1$, $\sigma^2 = 0$, and (5.21). Since $\bar{\Pi}^-(0+) > 0$ and $U(x) \geq x^2 \bar{\Pi}^-(x)$, (5.21) implies $\limsup_{x \downarrow 0} A(x)/x = \infty$. (5.66) also implies

$$\frac{{}^{(1)}\tilde{X}_{t_k}}{|\widetilde{\Delta X}_{t_k}^{(1)}|} \xrightarrow{P} \infty, \quad \text{as } k \rightarrow \infty,$$

so we have

$$\lim_{k \rightarrow \infty} P({}^{(1)}\tilde{X}_{t_k} \leq a |\widetilde{\Delta X}_{t_k}^{(1)}|) = 0 \quad \text{for some } a \in (0, 1).$$

Define b_k similarly as in (5.60):

$$b_k := \sup \left\{ x > 0 : \frac{x|A(x)| + U(x)}{x^2} > \frac{a^2}{56t_k} \right\}. \quad (5.74)$$

Then by the same calculation as in (5.60)–(5.62), we find, for large k ,

$$P({}^{(1)}\tilde{X}_{t_k} \leq a |\widetilde{\Delta X}_{t_k}^{(1)}|) \geq t_k \int_0^{\bar{\Pi}(b_k)} e^{-t_k v} dv / 2 = (1 - e^{-t_k \bar{\Pi}(b_k)}) / 2.$$

From this, we conclude that $t_k \bar{\Pi}(b_k) \rightarrow 0$. Take a subsequence $k' \rightarrow \infty$ if necessary so that

$$\frac{t_{k'} A(b_{k'})}{b_{k'}} \rightarrow A \quad \text{and} \quad \frac{t_{k'} U(b_{k'})}{b_{k'}^2} \rightarrow B, \quad (5.75)$$

where $B \geq 0$ and $|A| + B = a^2/56$.

Now $A \leq 0$ is not possible in (5.75). To see this, take a further subsequence of k' if necessary so that, for some functions $\bar{\Lambda}^\pm(x)$ and $B(x)$,

$$\lim_{k' \rightarrow \infty} t_{k'} \bar{\Pi}^\pm(x b_{k'}) = \bar{\Lambda}^\pm(x) \quad \text{and} \quad \lim_{k' \rightarrow \infty} \frac{t_{k'} U(x b_{k'})}{b_{k'}^2} = B(x)$$

at continuity points $x > 0$ of these functions. Let Λ be the measure having positive and negative tails $\bar{\Lambda}^\pm$. Then $\bar{\Lambda}(x) := \bar{\Lambda}^+(x) + \bar{\Lambda}^-(x) = 0$ for all $x \geq 1$. Fatou's lemma gives

$$\infty > B = \lim_{k' \rightarrow \infty} \frac{t_{k'} U(b_{k'})}{b_{k'}^2} = 2 \lim_{k' \rightarrow \infty} \int_0^1 y t_{k'} \bar{\Pi}(y b_{k'}) dy \geq 2 \int_0^1 y \bar{\Lambda}(y) dy,$$

and shows that the integral on the right is finite. This means that Λ is a Lévy measure on \mathbb{R} and by Kallenberg ([23], Theorem 15.14), as $k' \rightarrow \infty$ we have $(X_{t_{k'}} - t_{k'} \nu(b_{k'})) / b_{k'} \xrightarrow{D} Y'$, an infinitely divisible r.v. with canonical measure Λ . Since $\bar{\Lambda}(x) = 0$ for all $x \geq 1$, Y' has finite variance. Further, since $t_k \bar{\Pi}(b_k) \rightarrow 0$ we have $\lim_{k' \rightarrow \infty} t_{k'} \nu(b_{k'}) / b_{k'} = A$ (recall (5.6)). The Lévy–Itô decomposition can equivalently be written as

$$X_t = t\nu(b) + \sigma Z_t + X_t^{(S,b)} + X_t^{(B,b)}, \quad t \geq 0, \quad (5.76)$$

where $b > 0$, $X_t^{(S,b)}$ is the compensated small jump component of X , that is, having jumps less than or equal to b in modulus, and $X_t^{(B,b)}$ is the sum of jumps larger in modulus than b ; see, for example, Doney and Maller ([13], Lemma 6.1). Choose $b = b_k$ in (5.76), and notice that the sum of jumps larger in modulus than b_k is $o(b_k)$ as $k \rightarrow \infty$ because $t_k \bar{\Pi}(b_k) \rightarrow 0$. Also, $\sigma^2 = 0$. So we deduce

$$\frac{X_{t_{k'}}^{(S,b_{k'})} - t_{k'} \nu(b_{k'})}{b_{k'}} = \frac{X_{t_{k'}} - t_{k'} \nu(b_{k'})}{b_{k'}} + o_P(1) \xrightarrow{D} Y'. \quad (5.77)$$

From the inequality,

$$\frac{E(X_{t_{k'}}^{(S,b_{k'})})^2}{b_{k'}^2} \leq \frac{t_{k'} U(b_{k'})}{b_{k'}^2} \leq \frac{a^2}{56}$$

we see that $(X_{t_{k'}}^{(S,b_{k'})} / b_{k'})$ is uniformly integrable. Thus, we deduce from (5.77) that

$$\frac{E(X_{t_{k'}}^{(S,b_{k'})})}{b_{k'}} \rightarrow EY' + A.$$

The expectation on the left equals 0, so this implies $EY' = -A$. Now argue that

$$\lim_{k' \rightarrow \infty} P(X_{t_{k'}} \leq 0) = \lim_{k' \rightarrow \infty} P\left(\frac{X_{t_{k'}} - t_{k'} \nu(b_{k'})}{b_{k'}} \leq -\frac{t_{k'} \nu(b_{k'})}{b_{k'}}\right) = P(Y' \leq -A).$$

But since $Y' + A$ has mean 0 and finite variance, $P(Y' \leq -A) = P(Y' + A \leq 0) > 0$, in contradiction to (5.66). Thus, $A \leq 0$ is not possible.

We conclude that $A > 0$ and $B < \infty$. It follows from (5.75) that

$$\frac{A(b_{k'})}{\sqrt{U(b_{k'})\bar{\Pi}(b_{k'})}} \rightarrow \infty,$$

which implies (5.69).

Case (ii). Still assuming (5.66), suppose $\bar{\Pi}^-(0+) = 0$. (5.66) implies $P(X_{t_k} > 0) \rightarrow 1$, hence by Theorem 6, X is a subordinator and $A(x) \geq 0$ for all $x \geq 0$. Then

$$x^{-1}A(x) = x^{-1}\left(d_X + \int_0^x \bar{\Pi}^+(y) dy\right) \geq \int_0^1 \bar{\Pi}^+(xy) dy \rightarrow \infty, \quad \text{as } x \downarrow 0,$$

so we can define b_k by (5.74) and proceed as before to get $t_k \bar{\Pi}(b_k) \rightarrow 0$, and hence (5.69).

Conversely, in either cases (i) or (ii), we know (5.69) \implies (5.68), and (5.68) \implies (5.66) follows easily from the subsequential version of (5.24). \square

The following corollary to Theorem 9 is also proved in Theorem 4 of Maller [40].

Corollary 4. Assume $\bar{\Pi}(0+) > 0$. The following are equivalent:

- (i) $X_t \in SRS$ at 0;
- (ii) there are nonstochastic sequences $t_k \downarrow 0$ and $b_k > 0$, such that, as $k \rightarrow \infty$,

$$\frac{|X_{t_k}|}{b_k} \xrightarrow{P} 1; \tag{5.78}$$

(iii)

$$\sigma^2 = 0 \quad \text{and} \quad \limsup_{x \downarrow 0} \frac{|A(x)|}{\sqrt{\bar{\Pi}(x)U(x)}} = \infty; \tag{5.79}$$

(iv)

$$\limsup_{x \downarrow 0} \frac{x|A(x)|}{U(x)} = \infty. \tag{5.80}$$

Proof. Assume $\bar{\Pi}(0+) > 0$. First, $X_t \in SRS$ at 0 \implies (5.78) is obvious by definition.

(5.78) \implies (5.79) and (5.80): Let (5.78) hold with $t_k \downarrow 0$ and $b_k > 0$. Take a further subsequence $t_{k'} \downarrow 0$ if necessary so that $X_{t_{k'}}/b_{k'} \xrightarrow{D} Z'$. Z' is infinitely divisible by Lemma 4.1 of Maller and Mason [36]. Then $|Z'| = 1$ a.s., thus, as a bounded infinitely divisible random variable, Z' is degenerate at a constant which must be ± 1 . When $Z = +1$, $X \in PSRS$. Apply Theorem 9 to get (5.79) and (5.80). If $Z = -1$, $-X \in PSRS$. Then apply Theorem 9 to $-X$ to get (5.79) and (5.80) again.

(5.79) or (5.80) $\implies X_t \in SRS$ at 0: Let (5.79) or (5.80) hold. Then there is a sequence $x_k \downarrow 0$ as $k \rightarrow \infty$ such that $|A(x_k)| > 0$. By taking a further subsequence, we may assume

that $A(x_k) > 0$ for all k or $A(x_k) < 0$ for all k . Suppose the former; then (5.69) or (5.70) holds, so we get $X \in \text{PSRS}$ by Theorem 9. If the latter, then by applying Theorem 9 to $-X$, we get $X \in \text{NSRS}$. \square

5.5. Subsequential attraction to normality and dominance

We can also have subsequential convergence to normality, as $t \downarrow 0$. The next theorem gives an “uncentered” version of this. We describe (5.81) as “ $X \in D_{P0}(N)$ at 0”.

Theorem 10. *Assume $\sigma^2 > 0$ or $\bar{\Pi}(0+) = \infty$. Then there are nonstochastic sequences $t_k \downarrow 0$ and $b_k \downarrow 0$ such that, as $k \rightarrow \infty$,*

$$\frac{X_{t_k}}{b_k} \xrightarrow{D} N(0, 1); \quad (5.81)$$

iff

$$\limsup_{x \downarrow 0} \frac{U(x)}{x^2 \bar{\Pi}(x) + x|A(x)|} = \infty. \quad (5.82)$$

Proof. Both conditions hold when $\sigma^2 > 0$, so we can assume $\sigma^2 = 0$, thus, $\bar{\Pi}(0+) = \infty$. Let (5.82) hold and choose $x_k \downarrow 0$ such that

$$\frac{U(x_k)}{x_k^2 \bar{\Pi}(x_k)} \rightarrow \infty \quad \text{and} \quad \frac{U(x_k)}{x_k |A(x_k)|} \rightarrow \infty. \quad (5.83)$$

Then define

$$t_k = \min \left\{ \sqrt{\frac{x_k^2}{\bar{\Pi}(x_k)U(x_k)}}, \sqrt{\frac{x_k^3}{|A(x_k)|U(x_k)}} \right\}. \quad (5.84)$$

(If $A(x_k) = 0$ interpret the second component in (5.84) as $+\infty$.) Thus,

$$t_k \bar{\Pi}(x_k) \leq \sqrt{\frac{x_k^2 \bar{\Pi}(x_k)}{U(x_k)}} \rightarrow 0,$$

and since $\bar{\Pi}(0+) > 0$, we have $t_k \rightarrow 0$ as $k \rightarrow \infty$. Now let

$$b_k^2 = t_k U(x_k).$$

Since $\sigma^2 = 0$, $U(x_k) = 2 \int_0^{x_k} y \bar{\Pi}(y) dy \rightarrow 0$ as $k \rightarrow \infty$. Then $b_k \rightarrow 0$ as $k \rightarrow \infty$. Also

$$\frac{b_k^2}{x_k^2} = \min \left\{ \sqrt{\frac{U(x_k)}{x_k^2 \bar{\Pi}(x_k)}}, \sqrt{\frac{U(x_k)}{x_k |A(x_k)|}} \right\} \rightarrow \infty \quad (\text{by (5.83)}).$$

Given $x > 0$ choose k so large that $xb_k \geq x_k$. Then

$$t_k \bar{\Pi}(xb_k) \leq t_k \bar{\Pi}(x_k) \rightarrow 0,$$

and

$$\begin{aligned} \frac{t_k U(xb_k)}{b_k^2} &= 1 + \frac{t_k(U(xb_k) - U(x_k))}{b_k^2} = 1 + \frac{2t_k \int_{x_k}^{xb_k} y \bar{\Pi}(y) dy}{b_k^2} \\ &= 1 + O(t_k \bar{\Pi}(x_k)) = 1 + o(1). \end{aligned} \quad (5.85)$$

Also

$$\frac{t_k |A(x_k)|}{x_k} \leq \sqrt{\frac{x_k |A(x_k)|}{U(x_k)}} \rightarrow 0,$$

while

$$\begin{aligned} \frac{t_k |A(b_k)|}{b_k} &\leq o\left(\frac{t_k |A(x_k)|}{x_k}\right) + \frac{t_k \left| \int_{x_k}^{b_k} (\bar{\Pi}^+(y) - \bar{\Pi}^-(y)) dy \right|}{b_k} \\ &\leq o(1) + t_k \bar{\Pi}(x_k) = o(1). \end{aligned} \quad (5.86)$$

It follows from (5.85), (5.86) and the subsequential version of (5.48) that $X_{t_k}/b_k \xrightarrow{D} N(0, 1)$.

Conversely, if there is a $t_k \downarrow 0$ such that $X_{t_k}/b_k \xrightarrow{D} N(0, 1)$, then by the subsequential version of (5.48) we get (5.82). \square

Our final result in this section shows that a 2-sided version of (5.66) holds iff $X \in D_{P0}(N)$ at 0 or $X \in SRS$ at 0.

Theorem 11. *Assume $\bar{\Pi}(0+) = \infty$. Then the following are equivalent:*

$$\text{there is a nonstochastic sequence } t_k \downarrow 0 \quad (5.87)$$

$$\text{such that } \frac{|X_{t_k}|}{|\widetilde{\Delta X}_{t_k}^{(1)}|} \xrightarrow{P} \infty, \quad \text{as } k \rightarrow \infty;$$

$$\limsup_{x \downarrow 0} \frac{x|A(x)| + U(x)}{x^2 \bar{\Pi}(x)} = \infty; \quad (5.88)$$

$$(a) \limsup_{x \downarrow 0} \frac{U(x)}{x|A(x)| + x^2 \bar{\Pi}(x)} = +\infty, \quad \text{or} \quad (b) \limsup_{x \downarrow 0} \frac{x|A(x)|}{U(x)} = +\infty; \quad (5.89)$$

$$X \in D_{P0}(N) \cup SRS \quad \text{at } 0. \quad (5.90)$$

Proof. Assume $\bar{\Pi}(0+) = \infty$.

(5.87) \implies (5.88): Assume (5.87). Then just as in the proof of Theorem 8, we find $t_k \bar{\Pi}(b_k) \rightarrow 0$ as $k \rightarrow \infty$ where b_k satisfies (5.61). Thus, (5.88) holds.

(5.88) \implies (5.89) follows from Theorem 3 of Maller [40].

(5.89) \iff (5.90): follows from Theorem 10 and Corollary 4.

(5.90) \implies (5.87): (5.90) implies that there are $t_k \downarrow 0$, $b_k \downarrow 0$ such that $X_{t_k}/b_k \xrightarrow{D} N(0, 1)$ or $|X_{t_k}|/b_k \xrightarrow{P} 1$ as $k \rightarrow \infty$. Either of these implies $t_k \bar{\Pi}(b_k) \rightarrow 0$ as $k \rightarrow \infty$ and hence $\sup_{0 < s \leq t_k} |\Delta X_s|/b_k \xrightarrow{P} 0$ as $k \rightarrow \infty$. Thus, (5.87) holds. \square

Remark 5. (i) Theorems 10 and 11 have deep connections to generalised iterated logarithm laws for X_t as $t \downarrow 0$. It is shown in Theorem 3 of Maller [40] that (5.88) is equivalent to the existence of a nonstochastic function $B_t > 0$ such that

$$\limsup_{t \downarrow 0} \frac{|X_t|}{B_t} = 1 \quad \text{a.s.}$$

Maller [40] also gives a.s. equivalences for (5.46) and (5.89)(a). We hope to consider a.s. results related to those in Sections 3–5 elsewhere.

(ii) We note that in many conditions such as (5.88) and (5.89) we may replace the functions $A(x)$ and $U(x)$ in (5.4) and (5.5) by the functions $\nu(x)$ and $V(x)$ in (5.3). This is because

$$x|A(x) - \nu(x)| \leq x^2 \bar{\Pi}(x) \quad \text{and} \quad 0 \leq U(x) - V(x) = x^2 \bar{\Pi}(x), \quad x > 0.$$

But there is some advantage to working with the continuous functions $A(x)$ and $U(x)$, and sometimes it is essential, for example, in Theorem 5.

6. Related large time results

Most of the small time results derived herein have exact or close analogues for large times (i.e., allowing $t \rightarrow \infty$ rather than $t \downarrow 0$), some of them having been suggested by such analogies. In fact, many of the identities hold generally, for all $t > 0$; this is the case for all results in Section 2, as well as Lemmas 2 and 3. Some analogous large time results for Lévy processes can be found in Kevei and Mason [27], and Maller and Mason [37, 39], and we expect that others can be derived by straightforward modification of our small time methods. These would include compound Poisson processes as special cases.

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